CS 70 Discrete Mathematics and Probability Theory
Spring 2016 Rao and Walrand Discussion 9A Sol

Bayes' Rule: $P(A|B) = P(A \cap B)/P(B)$
Total probability rule: $P(A) = P(A \cap B) + P(A \cap \bar{B})$
Independence: $P(A \cap B) = P(A) * P(B)$

1. Probability Practice

1. If we put 5 math, 6 biology, 8 engineering, and 3 physics books on a bookshelf at random, what is the probability that all the math books are together?
2. A message source M of a digital communication system outputs a word of length 8 characters, with the characters drawn from the ternary alphabet $\{0,1,2\}$, and all such words are equally probable. What is the probability that M produces a word that looks like a byte (*i.e.*, no appearance of '2')?
3. If five numbers are selected at random from the set $\{1,2,3,\dots,20\}$ with replacement, what is the probability that their minimum is larger than 5?

Solution Sketch:

1. $\frac{18!5!}{22!} = \frac{1}{1463}$. The 18! comes from 18 "units": 3 physics books, 8 engineering books, 6 biology books and 1 block of math books. The 5! comes from number of ways to arrange the 5 math books within the same block. 22! is just the total number of ways to arrange the books.
2. $\left(\frac{2}{3}\right)^8 = \frac{256}{6561}$. This is just by independence.
3. $\left(\frac{3}{4}\right)^5 = \frac{243}{1024}$. For a single number, we can choose 6,7...20, so 15 valid outcomes out of 20 total outcomes => 3/4 probability. Then apply independence as in 2.2.

2. Disease diagnosis

You have a high fever and go to the doctor to identify the cause. 1% of the people have H1N1, 10% of the people have the flu, and 89% have neither. Assume that no person has both. Suppose that 100% of the H1N1 people have a high fever, 30% of the flu people have a high fever, and 2% of the people who have neither, have a high fever. Is it more likely that you have H1N1, the flu, or neither?

Let A be the event that the patient has H1N1, B be the event that the patient has Flu, and C be the event that the patient has neither. The event of having a fever is D. We want to compare $Pr(A|D)$, $Pr(B|D)$, and $Pr(C|D)$. We find each value using Bayes' rule.

$$Pr(A|D) = \frac{Pr(D|A)Pr(A)}{Pr(D|A)Pr(A) + Pr(D|B)Pr(B) + Pr(D|C)Pr(C)} = \frac{1 \times 0.01}{1 \times 0.01 + 0.1 \times 0.3 + 0.89 \times 0.02} = 0.173$$

$$Pr(B|D) = \frac{Pr(D|B)Pr(B)}{Pr(D|A)Pr(A) + Pr(D|B)Pr(B) + Pr(D|C)Pr(C)} = \frac{0.1 \times 0.3}{1 \times 0.01 + 0.1 \times 0.3 + 0.89 \times 0.02} = 0.519$$

$$Pr(C|D) = \frac{Pr(D|C)Pr(C)}{Pr(D|A)Pr(A) + Pr(D|B)Pr(B) + Pr(D|C)Pr(C)} = \frac{0.89 \times 0.02}{1 \times 0.01 + 0.1 \times 0.3 + 0.89 \times 0.02} = 0.308$$

So flu is the most likely.

3. Company Selection

Company A produces a particular device consisting of 10 components. Company A can either buy all the components from Company S or Company T, and then uses them to produce the devices without testing every individual component. After that, each device will be tested before leaving the factory. The device works only if every component works properly. Each working device can be sold for x dollars, but each non-working device must be thrown away. Products from Company S have a failure probability of $q = 0.01$ while Company T has a failure probability of $q/2$. However, every component from Company S costs $10 while it costs $30 from Company T. Should Company A build the device with components from Company S or Company T in order to maximize its expected profit per device? (Your answer will depend on x.)

Let W denote the event that a device works. Let R be the random variable denoting the profit.

$$\mathbf{E}[R] = P(W)\mathbf{E}[R|W] + P(W^C)\mathbf{E}[R|W^C].$$

Let's first consider the case when we use products from Company S. In this case, a device works with probability $P[W] = (1-q)^{10}$. The profit made on a working device is $x - 100$ dollars while a nonworking device has a profit of -100 dollars. That is, $\mathbf{E}[R|W] = x - 100$ and $\mathbf{E}[R|W^C] = -100$. Using R_S to denote the profit using components from Company S, the expected profit is:

$$\mathbf{E}[R_S] = (1-q)^{10}(x-100) + (1-(1-q)^{10})(-100) = (1-q)^{10}x - 100 = (0.99)^{10}x - 100.$$

If we use products from Company T. The device works with probability $P[W] = (1-q/2)^{10}$. The profit per working device is $\mathbf{E}[R|W] = x - 300$ dollars while the profit for a nonworking device is $\mathbf{E}[R|W^C] = -300$ dollars. The expected profit is:

$$\mathbf{E}[R_T] = (1-q/2)^{10}(x-300) + (1-(1-q/2)^{10})(-300) = (1-q/2)^{10}x - 300 = (0.995)^{10}x - 300.$$

To determine which Company should we use, we solve $\mathbf{E}[R_T] \geq \mathbf{E}[R_S]$, yielding $x \geq 200/[(0.995)^{10} - (0.99)^{10}] = 4280.1$. So for $x <$ \$4280.1 using products from Company S results in greater profit, while for $x >$ \$4280.1 more profit will be generated by using products from Company T.