Today.

Finish Counting.
Today.

Finish Counting.

...and then Professor Walrand.
But first..

This statement is a lie.
But first..

This statement is a lie. *Neither true nor false!*

---

```
def Turing(P):
    if Halts(P, P):
        while(True):
            pass
    else:
        return ...

Halt Program = ⇒ Turing Program. (P = ⇒ Q)

Turing("Turing")? = ⇒ No Turing program.

No Turing Program = ⇒ No halt program. (¬Q = ⇒ ¬P)

Program is text, so we can pass it to itself, or refer to self.
```
This statement is a lie. *Neither true nor false!*

Every person who doesn’t shave themselves is shaved by the barber.
But first..

This statement is a lie. *Neither true nor false!*

Every person who doesn’t shave themselves is shaved by the barber.

*Who shaves the barber?*
But first..

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    ...Text of Halt...

Halt Program $\implies$ Turing Program.
```
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Turing(“Turing”)?
But first..

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Halt Progam $\Rightarrow$ Turing Program. ($P \Rightarrow Q$)

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No Turing Program $\implies$ No halt program.
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...Text of Halt...

Halt Progam \implies\ Turing Program. (P \implies Q)

Turing(“Turing”)? Neither halts nor loops! \implies No Turing program.

No Turing Program \implies No halt program. (\neg Q \implies \neg P)
This statement is a lie. *Neither true nor false!*

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Halt Program $\implies$ Turing Program. $(P \implies Q)$

Turing(“Turing”)? Neither halts nor loops! $\implies$ No Turing program.

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Program is text, so we can pass it to itself,
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Halt Program $\Rightarrow$ Turing Program. ($P \Rightarrow Q$)

Turing("Turing")? Neither halts nor loops! $\Rightarrow$ No Turing program.

No Turing Program $\Rightarrow$ No halt program. ($\neg Q \Rightarrow \neg P$)

Program is text, so we can pass it to itself, or refer to self.
How many ways can Alice, Bob, and Eve split 5 dollars.
Splitting 5 dollars.

How many ways can Alice, Bob, and Eve split 5 dollars. Alice gets 3, Bob gets 1, Eve gets 1: \((A, A, A, B, E)\).
How many ways can Alice, Bob, and Eve split 5 dollars.

Alice gets 3, Bob gets 1, Eve gets 1: \((A, A, A, B, E)\).

Separate Alice’s dollars from Bob’s and then Bob’s from Eve’s.
How many ways can Alice, Bob, and Eve split 5 dollars.

Alice gets 3, Bob gets 1, Eve gets 1: \((A, A, A, B, E)\).

Separate Alice’s dollars from Bob’s and then Bob’s from Eve’s.

Five dollars are five stars: ⋆ ⋆ ⋆ ⋆ ⋆.
Splitting 5 dollars..

How many ways can Alice, Bob, and Eve split 5 dollars.

Alice gets 3, Bob gets 1, Eve gets 1: \((A, A, A, B, E)\).

Separate Alice’s dollars from Bob’s and then Bob’s from Eve’s.

Five dollars are five stars: ⋆ ⋆ ⋆ ⋆ ⋆.

Alice: 2, Bob: 1, Eve: 2.
Splitting 5 dollars..

How many ways can Alice, Bob, and Eve split 5 dollars.
Alice gets 3, Bob gets 1, Eve gets 1: \((A, A, A, B, E)\).
Separate Alice’s dollars from Bob’s and then Bob’s from Eve’s.
Five dollars are five stars: ⭐⭐⭐⭐⭐.
Alice: 2, Bob: 1, Eve: 2.
Stars and Bars: ⭐⭐|⭐|⭐⭐.
Splitting 5 dollars.

How many ways can Alice, Bob, and Eve split 5 dollars.
Alice gets 3, Bob gets 1, Eve gets 1: \((A, A, A, B, E)\).
Separate Alice’s dollars from Bob’s and then Bob’s from Eve’s.
Five dollars are five stars: * * * * *.
Alice: 2, Bob: 1, Eve: 2.
Stars and Bars: **|*|**.
Alice: 0, Bob: 1, Eve: 4.
How many ways can Alice, Bob, and Eve split 5 dollars.

Alice gets 3, Bob gets 1, Eve gets 1: \((A, A, A, B, E)\).

Separate Alice’s dollars from Bob’s and then Bob’s from Eve’s.

Five dollars are five stars: ⋆ ⋆ ⋆ ⋆ ⋆.

Alice: 2, Bob: 1, Eve: 2.
Stars and Bars: ⋆ ⋆ | ⋆ | ⋆ ⋆.

Alice: 0, Bob: 1, Eve: 4.
Stars and Bars: | ⋆ | ⋆ ⋆ ⋆.
Splitting 5 dollars..

How many ways can Alice, Bob, and Eve split 5 dollars.

Alice gets 3, Bob gets 1, Eve gets 1: \((A, A, A, B, E)\).

Separate Alice’s dollars from Bob’s and then Bob’s from Eve’s.

Five dollars are five stars: \(* * * * *\).

Alice: 2, Bob: 1, Eve: 2.
Stars and Bars: \(* * | * | **\).

Alice: 0, Bob: 1, Eve: 4.
Stars and Bars: \(| * | * * * *\).

Each split “is” a sequence of stars and bars.
How many ways can Alice, Bob, and Eve split 5 dollars.

Alice gets 3, Bob gets 1, Eve gets 1: \((A, A, A, B, E)\).

Separate Alice’s dollars from Bob’s and then Bob’s from Eve’s.

Five dollars are five stars: ★★★★★.

Alice: 2, Bob: 1, Eve: 2.
Stars and Bars: ★★|★|★★.

Alice: 0, Bob: 1, Eve: 4.
Stars and Bars: |★|★★★★.

Each split “is” a sequence of stars and bars.
Each sequence of stars and bars “is” a split.
Splitting 5 dollars..

How many ways can Alice, Bob, and Eve split 5 dollars.
Alice gets 3, Bob gets 1, Eve gets 1: (A, A, A, B, E).
Separate Alice’s dollars from Bob’s and then Bob’s from Eve’s.
Five dollars are five stars: ⭐⭐⭐⭐⭐.
Alice: 2, Bob: 1, Eve: 2.
Stars and Bars: ⭐⭐|⭐|⭐⭐.
Alice: 0, Bob: 1, Eve: 4.
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Five dollars are five stars: ⋆ ⋆ ⋆ ⋆ ⋆.
Alice: 2, Bob: 1, Eve: 2.
Stars and Bars: ⋆ ⋆ | ⋆ | ⋆ ⋆.
Alice: 0, Bob: 1, Eve: 4.
Stars and Bars: | ⋆ | ⋆ ⋆ ⋆.
Each split “is” a sequence of stars and bars.
Each sequence of stars and bars “is” a split.

Counting Rule: if there is a one-to-one mapping between two sets they have the same size!
Stars and Bars.

How many different 5 star and 2 bar diagrams?
Stars and Bars.

How many different 5 star and 2 bar diagrams?
| ⋆ | ⋆ ⋆ ⋆ ⋆.

7 positions in which to place the 2 bars.

Alice: 0; Bob 1; Eve: 4
| ⋆ | ⋆ ⋆ ⋆ ⋆.

Bars in first and third position.

Alice: 1; Bob 4; Eve: 0
⋆ | ⋆ ⋆ ⋆ ⋆ |

Bars in second and seventh position.

\[
\binom{7}{2} \text{ ways to do so and } \binom{7}{2} \text{ ways to split 5 dollars among 3 people.}
\]
Stars and Bars.

How many different 5 star and 2 bar diagrams?

| ⋆ | ⋆ ⋆ ⋆ ⋆ |

7 positions in which to place the 2 bars.
Stars and Bars.

How many different 5 star and 2 bar diagrams?

| * | * | * | * | *

7 positions in which to place the 2 bars.

Alice: 0; Bob 1; Eve: 4
Stars and Bars.

How many different 5 star and 2 bar diagrams?
| ⭐ | ⭐⭐⭐⭐ ⭐.
7 positions in which to place the 2 bars.

Alice: 0; Bob 1; Eve: 4
| ⭐ | ⭐⭐⭐⭐ ⭐.

(7\choose2) ways to do so and (7\choose2) ways to split 5 dollars among 3 people.
Stars and Bars.

How many different 5 star and 2 bar diagrams?

| ⋆  | ⋆ ⋆ ⋆ ⋆ |

7 positions in which to place the 2 bars.

-----

Alice: 0; Bob 1; Eve: 4

| ⋆  | ⋆ ⋆ ⋆ ⋆ |

Bars in first and third position.
Stars and Bars.

How many different 5 star and 2 bar diagrams?

| ⋆ | ⋆ ⋆ ⋆ ⋆ ⋆.

7 positions in which to place the 2 bars.

Alice: 0; Bob 1; Eve: 4

| ⋆ | ⋆ ⋆ ⋆ ⋆.

Bars in first and third position.

Alice: 1; Bob 4; Eve: 0
Stars and Bars.

How many different 5 star and 2 bar diagrams?
| ⋆ | ⋆ ⋆ ⋆ ⋆ |
7 positions in which to place the 2 bars.

Alice: 0; Bob 1; Eve: 4
| ⋆ | ⋆ ⋆ ⋆ ⋆ |
Bars in first and third position.

Alice: 1; Bob 4; Eve: 0
⋆ | ⋆ ⋆ ⋆ ⋆ |.
How many different 5 star and 2 bar diagrams?

| ⭐ | ⭐ ⭐ ⭐ ⭐ ⭐.

7 positions in which to place the 2 bars.

Alice: 0; Bob 1; Eve: 4

| ⭐ | ⭐ ⭐ ⭐ ⭐.

Bars in first and third position.

Alice: 1; Bob 4; Eve: 0

⭐ | ⭐ ⭐ ⭐ ⭐ |

Bars in second and seventh position.
Stars and Bars.

How many different 5 star and 2 bar diagrams?

| ⋆ | ⋆ ⋆ ⋆ ⋆ ⋆ |

7 positions in which to place the 2 bars.

Alice: 0; Bob 1; Eve: 4

| ⋆ | ⋆ ⋆ ⋆ ⋆ |

Bars in first and third position.

Alice: 1; Bob 4; Eve: 0

⋆ | ⋆ ⋆ ⋆ ⋆ |

Bars in second and seventh position.

\( \binom{7}{2} \) ways to do so and
Stars and Bars.

How many different 5 star and 2 bar diagrams?

\[
\begin{array}{c|c|c|c|c|c|c}
\star & \star & \star & \star & \star & & \\
\end{array}
\]

7 positions in which to place the 2 bars.

---

Alice: 0; Bob 1; Eve: 4

\[
\begin{array}{c|c|c|c|c|c|c}
\star & \star & \star & \star & \star & & \\
\end{array}
\]

Bars in first and third position.

Alice: 1; Bob 4; Eve: 0

\[
\begin{array}{c|c|c|c|c|c|c}
\star & \star & \star & \star & \star & & \\
\end{array}
\]

Bars in second and seventh position.

\[
\binom{7}{2}
\] ways to do so and

\[
\binom{7}{2}
\] ways to split 5 dollars among 3 people.
An alternative counting.
An alternative counting.

⋆ ⋆ ⋆ ⋆ ⋆
An alternative counting.

⋆ ⋆ ⋆ ⋆ ⋆ ⋆
An alternative counting.

Alternative: 6 places “in between” stars.
An alternative counting.

Alternative: 6 places “in between” stars.

Each selection of two places “in between” stars maps to an allocation of dollars to Alice, Bob, and Eve.
6 or 7???

An alternative counting.

⋆ ⋆ ⋆ ⋆ ⋆

Alternative: 6 places “in between” stars.

Each selection of two places “in between” stars maps to an allocation of dollars to Alice, Bob, and Eve.

Ways to choose two different places \( \binom{6}{2} \) plus
An alternative counting.

Alternative: 6 places “in between” stars.

Each selection of two places “in between” stars maps to an allocation of dollars to Alice, Bob, and Eve.

Ways to choose two different places \( \binom{6}{2} \) plus

ways to choose same place twice \( \binom{6}{1} = 6 \)
An alternative counting.

⋆ ⋆ ⋆ ⋆ ⋆

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Each selection of two places “in between” stars maps to an allocation of dollars to Alice, Bob, and Eve.

Ways to choose two different places $\binom{6}{2}$ plus

ways to choose same place twice $\binom{6}{1} = 6$
An alternative counting.

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Each selection of two places “in between” stars maps to an allocation of dollars to Alice, Bob, and Eve.

Ways to choose two different places \( \binom{6}{2} \) plus

ways to choose same place twice \( \binom{6}{1} = 6 \)

\( \binom{6}{2} + 6 \)
An alternative counting.

Alternative: 6 places “in between” stars.

Each selection of two places “in between” stars maps to an allocation of dollars to Alice, Bob, and Eve.

Ways to choose two different places $\binom{6}{2}$ plus ways to choose same place twice $\binom{6}{1} = 6$

$\binom{6}{2} + 6 = 21$. 
An alternative counting.

Alternative: 6 places “in between” stars.

Each selection of two places “in between” stars maps to an allocation of dollars to Alice, Bob, and Eve.

Ways to choose two different places \( \binom{6}{2} \) plus

ways to choose same place twice \( \binom{6}{1} = 6 \)

\[ \binom{6}{2} + 6 = 21. \]
An alternative counting.

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Ways to choose two different places $\binom{6}{2}$ plus ways to choose same place twice $\binom{6}{1} = 6$

$\binom{6}{2} + 6 = 21.$

$\binom{7}{2} = 21.$
An alternative counting.

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Ways to choose two different places $\binom{6}{2}$ plus ways to choose same place twice $\binom{6}{1} = 6$

$\binom{6}{2} + 6 = 21$.

$\binom{7}{2} = 21$.

For splitting among 4 people, this way becomes a mess.
6 or 7???

An alternative counting.

⋆ ⋆ ⋆ ⋆ ⋆

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$\binom{6}{2} + 6 = 21.$

$\binom{7}{2} = 21.$

For splitting among 4 people, this way becomes a mess.

$\binom{6}{3} + 2 \times \binom{6}{2} + \binom{6}{1}.$
An alternative counting.

Alternative: 6 places “in between” stars.

Each selection of two places “in between” stars maps to an allocation of dollars to Alice, Bob, and Eve.

Ways to choose two different places \( \binom{6}{2} \) plus ways to choose same place twice \( \binom{6}{1} = 6 \)

\( \binom{6}{2} + 6 = 21. \)

\( \binom{7}{2} = 21. \)

For splitting among 4 people, this way becomes a mess.

\( \binom{6}{3} + 2 \times \binom{6}{2} + \binom{6}{1}. \quad 20 + 30 + 6 = 56 \)
An alternative counting.

Alternative: 6 places “in between” stars.

Each selection of two places “in between” stars maps to an allocation of dollars to Alice, Bob, and Eve.

Ways to choose two different places $\binom{6}{2}$ plus ways to choose same place twice $\binom{6}{1} = 6$

$\binom{6}{2} + 6 = 21. \\
\binom{7}{2} = 21.$

For splitting among 4 people, this way becomes a mess.

$\binom{6}{3} + 2 * \binom{6}{2} + \binom{6}{1}$. $20 + 30 + 6 = 56$
An alternative counting.

⋆ ⋆ ⋆ ⋆ ⋆ ⋆

Alternative: 6 places “in between” stars.

Each selection of two places “in between” stars maps to an allocation of dollars to Alice, Bob, and Eve.

Ways to choose two different places \( \binom{6}{2} \) plus ways to choose same place twice \( \binom{6}{1} = 6 \)

\[ \binom{6}{2} + 6 = 21. \]

\[ \binom{7}{2} = 21. \]

For splitting among 4 people, this way becomes a mess.

\( \binom{6}{3} + 2 \cdot \binom{6}{2} + \binom{6}{1} \).

\[ 20 + 30 + 6 = 56 \]

\( \binom{8}{3} \).

\[ (8 \cdot 7 \cdot 6) / 6 = 56. \]
An alternative counting.

Alternative: 6 places “in between” stars.

Each selection of two places “in between” stars maps to an allocation of dollars to Alice, Bob, and Eve.

Ways to choose two different places \( \binom{6}{2} \) plus ways to choose same place twice \( \binom{6}{1} = 6 \)

\[ \binom{6}{2} + 6 = 21. \]

\[ \binom{7}{2} = 21. \]

For splitting among 4 people, this way becomes a mess.

\[ \binom{6}{3} + 2 \times \binom{6}{2} + \binom{6}{1}. \quad 20 + 30 + 6 = 56 \]

\[ \binom{8}{3}. \quad (8 \times 7 \times 6)/6 = 56. \]
An alternative counting.

Alternative: 6 places “in between” stars.

Each selection of two places “in between” stars maps to an allocation of dollars to Alice, Bob, and Eve.

Ways to choose two different places \( \binom{6}{2} \) plus ways to choose same place twice \( \binom{6}{1} = 6 \)

\( \binom{6}{2} + 6 = 21 \).

\( \binom{7}{2} = 21 \).

For splitting among 4 people, this way becomes a mess.

\( \binom{6}{3} + 2 \times \binom{6}{2} + \binom{6}{1} \). \quad 20 + 30 + 6 = 56

\( \binom{8}{3} \). \quad (8\times7\times6)/6 = 56 \).
Stars and Bars.

Ways to add up $n$ numbers to sum to $k$?
Stars and Bars.

Ways to add up $n$ numbers to sum to $k$? or

“$k$ from $n$ with replacement where order doesn’t matter.”
Stars and Bars.

Ways to add up $n$ numbers to sum to $k$? or

"$k$ from $n$ with replacement where order doesn’t matter.”

In general, $k$ stars $n - 1$ bars.

$$\ast \ast \mid \ast \mid \cdots \mid \ast \ast.$$
Stars and Bars.

Ways to add up $n$ numbers to sum to $k$? or

“$k$ from $n$ with replacement where order doesn’t matter.”

In general, $k$ stars $n−1$ bars.

$$\star \star | \star | \cdots | \star \star.$$  

$n + k − 1$ positions from which to choose $n − 1$ bar positions.
Stars and Bars.

Ways to add up $n$ numbers to sum to $k$? or

“$k$ from $n$ with replacement where order doesn’t matter.”

In general, $k$ stars $n-1$ bars.

\[
\begin{array}{c|c|\cdots|c}
\star & \star & \cdots & \star \\
\end{array}
\]

$n + k - 1$ positions from which to choose $n - 1$ bar positions.

\[
\binom{n+k-1}{n-1}
\]
Stars and Bars.

Ways to add up \( n \) numbers to sum to \( k \)? or

“\( k \) from \( n \) with replacement where order doesn’t matter.”

In general, \( k \) stars \( n - 1 \) bars.

\[
\begin{align*}
\star \star & | \star | \cdots | \star \\
\star & \star
\end{align*}
\]

\( n + k - 1 \) positions from which to choose \( n - 1 \) bar positions.

\[
\binom{n + k - 1}{n - 1}
\]

Or: \( k \) unordered choices from set of \( n \) possibilities with replacement.

Sample with replacement where order doesn’t matter.
Quick review of the basics.

**First rule:** $n_1 \times n_2 \cdots \times n_3$. 
Quick review of the basics.

**First rule:** $n_1 \times n_2 \cdots \times n_3$.

$k$ Samples with replacement from $n$ items: $n^k$. 
Quick review of the basics.

**First rule:** \( n_1 \times n_2 \cdots \times n_3 \).

\( k \) Samples with replacement from \( n \) items: \( n^k \).
Sample without replacement: \( \frac{n!}{(n-k)!} \).
Quick review of the basics.

**First rule:** \( n_1 \times n_2 \cdots \times n_3. \)

\( k \) Samples with replacement from \( n \) items: \( n^k. \)
Sample without replacement: \( \frac{n!}{(n-k)!} \)

**Second rule:** when order doesn’t matter divide..when possible.
Quick review of the basics.

**First rule:** $n_1 \times n_2 \cdots \times n_3$.

$k$ Samples with replacement from $n$ items: $n^k$.
Sample without replacement: $\frac{n!}{(n-k)!}$

**Second rule:** when order doesn’t matter divide..when possible.
Sample without replacement and order doesn’t matter: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.
“$n$ choose $k$”
Quick review of the basics.

**First rule:** $n_1 \times n_2 \cdots \times n_3$.

$k$ Samples with replacement from $n$ items: $n^k$.
Sample without replacement: $\frac{n!}{(n-k)!}$

**Second rule:** when order doesn’t matter divide..when possible.
Sample without replacement and order doesn’t matter: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.
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**One-to-one rule:** equal in number if one-to-one correspondence.
Quick review of the basics.

**First rule:** \( n_1 \times n_2 \cdots \times n_3. \)

\( k \) Samples with replacement from \( n \) items: \( n^k. \)
Sample without replacement: \( \frac{n!}{(n-k)!} \)

**Second rule:** when order doesn’t matter divide..when possible.
Sample without replacement and order doesn’t matter: \( \binom{n}{k} = \frac{n!}{(n-k)!k!}. \)

“\( n \) choose \( k \)”

**One-to-one rule:** equal in number if one-to-one correspondence.
Sample with replacement and order doesn’t matter: \( \binom{k+n-1}{n-1}. \)
Balls in bins.

\[ \text{balls in } n \text{ bins} \equiv \text{k samples from } n \text{ possibilities.} \]

"indistinguishable balls" \equiv "order doesn't matter"

"only one ball in each bin" \equiv "without replacement"

5 balls into 10 bins
5 samples from 10 possibilities with replacement
Example: 5 digit numbers.
5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement
Example: Poker hands.
5 indistinguishable balls into 3 bins
5 samples from 3 possibilities with replacement and no order
Dividing 5 dollars among Alice, Bob and Eve.
Balls in bins.

“$k$ Balls in $n$ bins” $\equiv$ “$k$ samples from $n$ possibilities.”
Balls in bins.

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Balls in bins.

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5 samples from 10 possibilities with replacement
   Example: 5 digit numbers.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement
   Example: Poker hands.
Balls in bins.

\[0\quad 1 \quad \cdots \quad n\]

“\(k\) Balls in \(n\) bins” \(\equiv\) “\(k\) samples from \(n\) possibilities.”

“indistinguishable balls” \(\equiv\) “order doesn’t matter”

“only one ball in each bin” \(\equiv\) “without replacement”

5 balls into 10 bins
5 samples from 10 possibilities with replacement

Example: 5 digit numbers.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement

Example: Poker hands.

5 indistinguishable balls into 3 bins
Balls in bins.

\[ \begin{array}{ccc|c|c} \hline \text{0} & \text{1} & \cdots & \text{n} \\ \hline \end{array} \]

“\(k\) Balls in \(n\) bins” \(\equiv\) “\(k\) samples from \(n\) possibilities.”

“indistinguishable balls” \(\equiv\) “order doesn’t matter”

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5 balls into 10 bins
5 samples from 10 possibilities with replacement

Example: 5 digit numbers.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement

Example: Poker hands.

5 indistinguishable balls into 3 bins
5 samples from 3 possibilities with replacement and no order
Balls in bins.

\[
\begin{array}{c|c|c|c}
1 & \cdot & \cdot & n \\
0 & 1 & \cdot & n \\
\end{array}
\]

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Dividing 5 dollars among Alice, Bob and Eve.
Sum Rule

Two indistinguishable jokers in 54 card deck. How many 5 card poker hands?
Sum Rule

Two indistinguishable jokers in 54 card deck. How many 5 card poker hands? **Sum rule: Can sum over disjoint sets.**
Sum Rule

Two indistinguishable jokers in 54 card deck. How many 5 card poker hands?

**Sum rule: Can sum over disjoint sets.**
Two indistinguishable jokers in 54 card deck. How many 5 card poker hands?

**Sum rule:** Can sum over disjoint sets.

No jokers

\[
\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.
\]

Two distinguishable jokers in 54 card deck. How many 5 card poker hands?

Choose 4 cards plus one of 2 jokers!

\[
\binom{52}{5} + 2 \cdot \binom{52}{4} + \binom{52}{3}.
\]

Wait a minute! The same as choosing 5 cards from 54.

**Theorem:**

\[
\binom{54}{5} = \binom{52}{5} + 2 \cdot \binom{52}{4} + \binom{52}{3}.
\]
Two indistinguishable jokers in 54 card deck.
How many 5 card poker hands?

**Sum rule:** Can sum over disjoint sets.
No jokers “exclusive” or One Joker

\[
\binom{52}{5} + \binom{52}{4}.
\]

Two distinguishable jokers in 54 card deck.
How many 5 card poker hands?

Choose 4 cards plus one of 2 jokers!

\[
\binom{52}{5} + 2 \times \binom{52}{4} + \binom{52}{3}.
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Wait a minute!

Same as choosing 5 cards from 54

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Sum rule: Can sum over disjoint sets.

No jokers “exclusive” or One Joker “exclusive” or Two Jokers

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\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.
\]
Sum Rule

Two indistinguishable jokers in 54 card deck. How many 5 card poker hands?

**Sum rule:** Can sum over disjoint sets.

No jokers “exclusive” or One Joker “exclusive” or Two Jokers

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\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.
\]

Two distinguishable jokers in 54 card deck.
**Sum Rule**

Two indistinguishable jokers in 54 card deck. How many 5 card poker hands?

**Sum rule:** *Can sum over disjoint sets.*

No jokers “exclusive” or One Joker “exclusive” or Two Jokers

$$\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.$$ 

Two distinguishable jokers in 54 card deck. How many 5 card poker hands?
Two indistinguishable jokers in 54 card deck.
How many 5 card poker hands?

**Sum rule:** Can sum over disjoint sets.
No jokers “exclusive” or One Joker “exclusive” or Two Jokers

\[
\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.
\]

Two distinguishable jokers in 54 card deck.
How many 5 card poker hands?

\[
\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.
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Wait a minute!
Same as choosing 5 cards from 54
or \[
\binom{54}{5}.
\]

Theorem:
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\binom{54}{5} = \binom{52}{5} + 2 \cdot \binom{52}{4} + \binom{52}{3}.
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Two indistinguishable jokers in 54 card deck. How many 5 card poker hands?

**Sum rule:** Can sum over disjoint sets.
No jokers “exclusive” or One Joker “exclusive” or Two Jokers

\[
\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.
\]

Two distinguishable jokers in 54 card deck. How many 5 card poker hands? Choose 4 cards plus one of 2 jokers!

\[
\binom{52}{5} + 2 \times \binom{52}{4} +
\]
Two indistinguishable jokers in 54 card deck.
How many 5 card poker hands?

**Sum rule:** Can sum over disjoint sets.

No jokers “exclusive” or One Joker “exclusive” or Two Jokers

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Two distinguishable jokers in 54 card deck.
How many 5 card poker hands? **Choose 4 cards plus one of 2 jokers!**

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How many 5 card poker hands? Choose 4 cards plus one of 2 jokers!

\[ \binom{52}{5} + 2 \times \binom{52}{4} + \binom{52}{3} \]

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Two distinguishable jokers in 54 card deck.
How many 5 card poker hands? Choose 4 cards plus one of 2 jokers!

\[
\binom{52}{5} + 2 \times \binom{52}{4} + \binom{52}{3}
\]

Wait a minute! Same as choosing 5 cards from 54 or
Sum Rule

Two indistinguishable jokers in 54 card deck. How many 5 card poker hands?

**Sum rule: Can sum over disjoint sets.**

No jokers “exclusive” or One Joker “exclusive” or Two Jokers

\[
\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.
\]

Two distinguishable jokers in 54 card deck. How many 5 card poker hands? Choose 4 cards plus one of 2 jokers!

\[
\binom{52}{5} + 2 \times \binom{52}{4} + \binom{52}{3}
\]

Wait a minute! Same as choosing 5 cards from 54 or

\[
\binom{54}{5}
\]

**Theorem:** \(\binom{54}{5}\)
Sum Rule

Two indistinguishable jokers in 54 card deck. How many 5 card poker hands?

**Sum rule:** Can sum over disjoint sets.

No jokers “exclusive” or One Joker “exclusive” or Two Jokers

\[
\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.
\]

Two distinguishable jokers in 54 card deck. How many 5 card poker hands? Choose 4 cards plus one of 2 jokers!

\[
\binom{52}{5} + 2 \times \binom{52}{4} + \binom{52}{3}
\]

Wait a minute! Same as choosing 5 cards from 54 or

\[
\binom{54}{5}
\]

**Theorem:** \( \binom{54}{5} = \binom{52}{5} + 2 \times \binom{52}{4} + \binom{52}{3} \).
Combinatorial Proofs.

**Theorem:** \( \binom{n}{k} = \binom{n}{n-k} \)
Theorem: \( \binom{n}{k} = \binom{n}{n-k} \)

Proof: How many subsets of size \( k \)?
Combinatorial Proofs.

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How many subsets of size \( k \)?
Combinatorial Proofs.

Theorem: \( \binom{n}{k} = \binom{n}{n-k} \)

Proof: How many subsets of size \( k \)? \( \binom{n}{k} \)

How many subsets of size \( k \)?
Choose a subset of size \( n - k \)
Combinatorial Proofs.

**Theorem:** \( \binom{n}{k} = \binom{n}{n-k} \)

**Proof:** How many subsets of size \( k \)? \( \binom{n}{k} \)

How many subsets of size \( k \)?
Choose a subset of size \( n-k \)
and what’s left out
Combinatorial Proofs.

**Theorem:** \( \binom{n}{k} = \binom{n}{n-k} \)

**Proof:** How many subsets of size \( k \)? \( \binom{n}{k} \)

How many subsets of size \( k \)?
Choose a subset of size \( n - k \)
    and what’s left out is a subset of size \( k \).
Combinatorial Proofs.

**Theorem:** \( \binom{n}{k} = \binom{n}{n-k} \)

**Proof:** How many subsets of size \( k \)? \( \binom{n}{k} \)

How many subsets of size \( k \)?
Choose a subset of size \( n-k \)
   and what’s left out is a subset of size \( k \).
Choosing a subset of size \( k \) is same
Combinatorial Proofs.

**Theorem:** \( \binom{n}{k} = \binom{n}{n-k} \)

**Proof:** How many subsets of size \( k \)? \( \binom{n}{k} \)

How many subsets of size \( k \)?
Choose a subset of size \( n - k \)
and what’s left out is a subset of size \( k \).
Choosing a subset of size \( k \) is same
as choosing \( n - k \) elements to not take.
Theorem: \( \binom{n}{k} = \binom{n}{n-k} \)

Proof: How many subsets of size \( k \)? \( \binom{n}{k} \)

How many subsets of size \( k \)?
Choose a subset of size \( n-k \)
    and what’s left out is a subset of size \( k \).
Choosing a subset of size \( k \) is same
    as choosing \( n-k \) elements to not take.
\( \implies \binom{n}{n-k} \) subsets of size \( k \).
Theorem: \( \binom{n}{k} = \binom{n}{n-k} \)

Proof: How many subsets of size \( k \)? \( \binom{n}{k} \)

How many subsets of size \( k \)?
Choose a subset of size \( n-k \)
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Choosing a subset of size \( k \) is same
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\( \implies \binom{n}{n-k} \) subsets of size \( k \).
Pascal’s Triangle

Row \( n \): coefficients of \((1 + x)^n\) = \((1 + x)(1 + x) \cdots (1 + x)\).

Foil (4 terms) on steroids: \(2^n\) terms: choose 1 or \(x\) from each factor of \((1 + x)\).

Simplify: collect all terms corresponding to \(x^k\).

Coefficient of \(x^k\) is \(\binom{n}{k}\): choose \(k\) factors where \(x\) is in product.

\[
\begin{align*}
\binom{0}{0} & = 1 \\
\binom{1}{0} & = 1 \\
\binom{1}{1} & = 1 \\
\binom{2}{0} & = 1 \\
\binom{2}{1} & = 2 \\
\binom{2}{2} & = 1 \\
\binom{3}{0} & = 1 \\
\binom{3}{1} & = 3 \\
\binom{3}{2} & = 3 \\
\binom{3}{3} & = 1
\end{align*}
\]

Pascal’s rule => \(\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}\).
Pascal’s Triangle

\[
\begin{array}{c|c|c|c|c|c|c}
& & & & & & \\
& 1 & & & & & \\
& 1 & 1 & & & & \\
& 1 & 2 & 1 & & & \\
& 1 & 3 & 3 & 1 & & \\
& 1 & 4 & 6 & 4 & 1 & \\
\end{array}
\]

Row \( n \): coefficients of \((1 + x)^n\).

Foil (4 terms) on steroids: \(2^n\) terms: choose 1 or \(x\) from each factor of \((1 + x)^n\).

Simplify: collect all terms corresponding to \(x^k\).

Coefficient of \(x^k\) is \(\binom{n}{k}\): choose \(k\) factors where \(x\) is in product.

\[
\begin{align*}
\binom{0}{0} & \quad \binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} & \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} & \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\end{align*}
\]

Pascal’s rule \( \Rightarrow \):

\[
\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}
\]
Pascal’s Triangle

\begin{align*}
1 \\
1 & 1 \\
1 & 2 & 1 \\
\end{align*}

Row \( n \): coefficients of \((1 + x)^n = (1 + x)(1 + x) \cdots (1 + x)\).

\text{Foil (4 terms)}

\text{on steroids: 2 terms: choose 1 or } x \text{ from each factor of } (1 + x).

\text{Simplify: collect all terms corresponding to } x^k.

\text{Coefficient of } x^k \text{ is } \binom{n}{k}: \text{choose } k \text{ factors where } x \text{ is in product}.

\begin{align*}
\binom{0}{0} & \quad \binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} & \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} & \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\end{align*}

\text{Pascal’s rule} \Rightarrow \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.
Pascal’s Triangle

1
1 1
1 2 1
1 3 3 1
Row \( n \): coefficients of \((1 + x)^n\) = \((1 + x)(1 + x) \cdots (1 + x)\).

Foil (4 terms) on steroids: \(2^n\) terms: choose 1 or \(x\) from each factor of \((1 + x)\).

Simplify: collect all terms corresponding to \(x^k\).

Coefficient of \(x^k\) is \(\binom{n}{k}\): choose \(k\) factors where \(x\) is in product.

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 & 0 \\
1 & 3 & 3 & 1 & 0 & 0 \\
1 & 4 & 6 & 4 & 1 & 0 \\
\end{array}
\]

Pascal’s rule: \(\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}\).
Pascal’s Triangle

\[
\begin{array}{cccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
\end{array}
\]

Row \(n\): coefficients of \((1 + x)^n\) = \((1 + x)(1 + x) \cdots (1 + x)\).

Foil (4 terms): \(2^n\) terms: choose 1 or \(x\) from each factor of \((1 + x)(1 + x) \cdots (1 + x)\).

Simplify: collect all terms corresponding to \(x^k\).

Coefficient of \(x^k\) is \(\binom{n}{k}\): choose \(k\) factors where \(x\) is in product.

\[
\begin{array}{cccc}
\binom{0}{0} & \binom{1}{0} & \binom{1}{1} & \binom{2}{0} & \binom{2}{1} & \binom{2}{2} \\
\binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} \\
\end{array}
\]

Pascal’s rule ⇒ \(\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}\).
Pascal’s Triangle

Row $n$: coefficients of $(1 + x)^n = (1 + x)(1 + x) \cdots (1 + x)$.

Foil (4 terms) on steroids: $2^n$ terms: choose 1 or $x$ from each factor of $(1 + x)$.

Simplify: collect all terms corresponding to $x^k$.

Coefficient of $x^k$ is $\binom{n}{k}$: choose $k$ factors where $x$ is in product.

$\begin{array}{cccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
\end{array}$

Pascal’s rule $\Rightarrow \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$. 
Pascal’s Triangle

Row \( n \): coefficients of \((1 + x)^n = (1 + x)(1 + x) \cdots (1 + x)\).
Pascal’s Triangle

Row $n$: coefficients of $(1 + x)^n = (1 + x)(1 + x) \cdots (1 + x)$.

Foil (4 terms)
Pascal’s Triangle

\[
\begin{align*}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1
\end{align*}
\]

Row \( n \): coefficients of \((1 + x)^n = (1 + x)(1 + x) \cdots (1 + x)\).

Foil (4 terms) on steroids:
Pascal’s Triangle

Row $n$: coefficients of $(1 + x)^n = (1 + x)(1 + x) \cdots (1 + x)$.

Foil (4 terms) on steroids:

$2^n$ terms:
Pascal’s Triangle

\[
\begin{array}{c}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
\end{array}
\]

Row \( n \): coefficients of \((1 + x)^n = (1 + x)(1 + x) \cdots (1 + x)\).

Foil (4 terms) on steroids:

2\(^n\) terms: choose 1 or \(x\) from each factor of \((1 + x)\).
Pascal’s Triangle

Row $n$: coefficients of $(1 + x)^n = (1 + x)(1 + x) \cdots (1 + x)$.

Foil (4 terms) on steroids:
$2^n$ terms: choose 1 or $x$ from each factor of $(1 + x)$. 

Pascal’s Triangle

Row \( n \): coefficients of \((1 + x)^n = (1 + x)(1 + x) \cdots (1 + x)\).

Foil (4 terms) on steroids:
- \(2^n\) terms: choose 1 or \(x\) from each factor of \((1 + x)\).

Simplify: collect all terms corresponding to \(x^k\).
Pascal’s Triangle

\[
\begin{array}{cccccc}
& & & & & 1 \\
& & & & 1 & 1 \\
& & & 1 & 2 & 1 \\
& & 1 & 3 & 3 & 1 \\
& 1 & 4 & 6 & 4 & 1 \\
\end{array}
\]

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Foil (4 terms) on steroids:
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Simplify: collect all terms corresponding to \(x^k\).
Coefficient of \(x^k\) is \(\binom{n}{k}\): choose \(k\) factors where \(x\) is in product.
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Foil (4 terms) on steroids:
- \(2^n\) terms: choose 1 or \(x\) from each factor of \((1 + x)\).

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\[
\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \\
\binom{1}{1}
\end{array}
\]
Pascal’s Triangle

\[
\begin{array}{cccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
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1 & 4 & 6 & 4 & 1 \\
\end{array}
\]

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Foil (4 terms) on steroids:
- \(2^n \) terms: choose 1 or \(x\) from each factor of \((1 + x)\).

Simplify: collect all terms corresponding to \(x^k\).
- Coefficient of \(x^k\) is \(\binom{n}{k}\): choose \(k\) factors where \(x\) is in product.

\[
\begin{array}{cccc}
\binom{0}{0} & \binom{1}{1} & \binom{2}{2} \\
\binom{1}{0} & \binom{1}{1} & \binom{2}{2} \\
\binom{2}{0} & \binom{1}{1} & \binom{2}{2} \\
\end{array}
\]
Pascal’s Triangle

Row $n$: coefficients of $(1 + x)^n = (1 + x)(1 + x) \cdots (1 + x)$.

Foil (4 terms) on steroids:

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Simplify: collect all terms corresponding to $x^k$.

Coefficient of $x^k$ is $\binom{n}{k}$: choose $k$ factors where $x$ is in product.
Pascal’s Triangle

\[
\begin{array}{cccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
\end{array}
\]

Row \( n \): coefficients of \((1 + x)^n = (1 + x)(1 + x) \cdots (1 + x)\).

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\binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} \\
\end{array}
\]

Pascal’s rule \(\implies\) \(\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}\).
Combinatorial Proofs.

**Theorem:** \( \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \).

**Proof:** How many size \( k \) subsets of \( n+1 \)?

How many size \( k \) subsets of \( n+1 \)? How many contain the first element? Choose first element, need to choose \( k-1 \) more from remaining \( n \) elements.

How many don't contain the first element? Need to choose \( k \) elements from remaining \( n \) elts.

So, \( \binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k} \).
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Chose first element,
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**Proof:** How many size $k$ subsets of $n+1$? $\binom{n+1}{k}$.

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$\Rightarrow \binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$. 
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\[ \{1, \ldots, i, \ldots, n\} \]

Must choose \( k - 1 \) elements from \( n - i \) remaining elements.
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which is also \( \binom{n+1}{k} \).

\[ \square \]
Binomial Theorem: $x = 1$

Theorem: $2^n = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{0}$
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Construct a subset with sequence of $n$ choices:
- element $i$ is in or is not in the subset: 2 poss.
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Simple Inclusion/Exclusion

**Sum Rule:** For disjoint sets $S$ and $T$, $|S \cup T| = |S| + |T|$

Used to reason about all subsets

by adding number of subsets of size 1, 2, 3, ...
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**Sum Rule:** For disjoint sets $S$ and $T$, $|S \cup T| = |S| + |T|$

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Also reasoned about subsets that contained or didn’t contain an element. (E.g., first element, first $i$ elements.)
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**Inclusion/Exclusion Rule:** For any $S$ and $T$,

$|S \cup T| = |S| + |T| - |S \cap T|$. 

Example:

How many 10-digit phone numbers have 7 as their first or second digit?

Let $S$ be phone numbers with 7 as first digit.

$|S| = 10^9$

Let $T$ be phone numbers with 7 as second digit.

$|T| = 10^9$

Let $S \cap T$ be phone numbers with 7 as first and second digit.

$|S \cap T| = 10^8$

Answer:

$|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$. 
Simple Inclusion/Exclusion

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Answer: $|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$. 
Summary.

First Rule of counting:

Objects from a sequence of choices: \( n_1 \times n_2 \times \cdots \times n_k \) objects.

Second Rule of counting:

If order does not matter. Count with order. Divide by number of orderings/sorted object. Typically: \( \binom{n}{k} \).

Stars and Bars:

Sample \( k \) objects with replacement from \( n \).

Order doesn't matter. Typically: \( \binom{n+k-1}{k-1} \).

Inclusion/Exclusion: two sets of objects. Add number of each subtract intersection of sets.

Sum Rule: If disjoint just add.

Combinatorial Proofs: Identity from counting same in two ways.

Pascal's Triangle Example:

\( \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \).

RHS: Number of subsets of \( n+1 \) items size \( k \).

LHS: \( \binom{n}{k-1} \) counts subsets of \( n+1 \) items with first item. \( \binom{n}{k} \) counts subsets of \( n+1 \) items without first item. Disjoint – so add!
Summary.

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First Rule of counting: Objects from a sequence of choices:
\[ n_i \text{ possibilities for } i\text{th choice.} \]
Summary.

First Rule of counting: Objects from a sequence of choices:
- $n_i$ possibilities for $i$th choice.
- $n_1 \times n_2 \times \cdots \times n_k$ objects.
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Count with order. Divide by number of orderings/sorted object.

Typically: $\binom{n}{k}$.

Stars and Bars: Sample $k$ objects with replacement from $n$.
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