Modeling Uncertainty: Probability Space
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1. Key Points
2. Random Experiments
3. Probability Space
Key Points

- Uncertainty does not mean "nothing is known".
- How to best make decisions under uncertainty?
  - Buy stocks
  - Detect signals (transmitted bits, speech, images, radar, diseases, etc.)
  - Control systems (Internet, airplane, robots, self-driving cars, schedule surgeries in a hospital, etc.)
- How to best use 'artificial' uncertainty?
  - Play games of chance
  - Design randomized algorithms.
- Probability models knowledge about uncertainty.
  - Discovers best way to use that knowledge in making decisions.
Key Points

- Uncertainty does not mean “nothing is known”
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▶ Uncertainty does not mean “nothing is known”
▶ How to best make decisions under uncertainty?

Examples:
- Buying stocks
- Detecting signals (transmitted bits, speech, images, radar, diseases, etc.)
- Controlling systems (Internet, airplane, robots, self-driving cars, schedule surgeries in a hospital, etc.)
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  - Models knowledge about uncertainty
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▶ Probability
  ▶ Models knowledge about uncertainty
  ▶ Discovers best way to use that knowledge in making decisions
The Magic of Probability

Uncertainty: vague, fuzzy, confusing, scary, hard to think about.

Probability: a precise, unambiguous, simple way to think about uncertainty.

Our mission: help you discover the serenity of Probability, i.e., enable you to think clearly about uncertainty.

Your cost: focused attention and practice on examples and problems.
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![Uncertainty = Fear](image_url)
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Uncertainty = Fear

Probability = Serenity
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Random Experiment: Flip one Fair Coin

Flip a fair coin:

▶ Possible outcomes: Heads (H) and Tails (T)

▶ Likelihoods: H: 50% and T: 50%
Random Experiment: Flip one Fair Coin

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Random Experiment: Flip one Fair Coin

Flip a **fair** coin: *(One flips or tosses a coin)*
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Flip a fair coin: (One flips or tosses a coin)

- Possible outcomes:

  ▶ H: 50% and T: 50%
Random Experiment: Flip one Fair Coin

Flip a fair coin: (One flips or tosses a coin)

Possible outcomes: Heads ($H$)
Random Experiment: Flip one Fair Coin

Flip a **fair** coin: *(One flips or tosses a coin)*

- Possible outcomes: Heads (*H*) and Tails (*T*)
Random Experiment: Flip one Fair Coin

Flip a **fair** coin: *(One flips or tosses a coin)*

- **Possible outcomes**: Heads (*H*) and Tails (*T*)
  *(One flip yields either ‘heads’ or ‘tails’)*
Random Experiment: Flip one Fair Coin

Flip a *fair* coin: (*One flips or tosses a coin*)

- Possible outcomes: Heads \( (H) \) and Tails \( (T) \)
  (*One flip yields either ‘heads’ or ‘tails’.*)
- Likelihoods:
Random Experiment: Flip one Fair Coin

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- Likelihoods: $H : 50\%$ and $T : 50\%$
Random Experiment: Flip one Fair Coin

Flip a fair coin:

What do we mean by the likelihood of tails is 50%?

Two interpretations:
- Single coin flip: 50% chance of 'tails' [subjectivist]
  - Willingness to bet on the outcome of a single flip
- Many coin flips: About half yield 'tails' [frequentist]
  - Makes sense for many flips

Question:
Why does the fraction of tails converge to the same value every time?

Statistical Regularity! Deep!
Random Experiment: Flip one Fair Coin

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Flip a fair coin:
Random Experiment: Flip one Fair Coin

Flip a **fair** coin: model
Random Experiment: Flip one Fair Coin

Flip a fair coin: model

Physical Experiment

Probability Model

\[ \Omega = \{H, T\} \]

\[ Pr[H] = 0.5, \quad Pr[T] = 0.5. \]
Random Experiment: Flip one Fair Coin

Flip a *fair* coin: model

The physical experiment is complex.
Random Experiment: Flip one Fair Coin

Flip a **fair** coin: model

- The physical experiment is complex. (Shape, density, initial momentum and position, ...)

![Physical Experiment](image.png) ![Probability Model](probability_model.png)
Random Experiment: Flip one Fair Coin

Flip a fair coin: model

- The physical experiment is complex. (Shape, density, initial momentum and position, ...)
- The Probability model is simple:
Random Experiment: Flip one Fair Coin

Flip a **fair** coin: model

- The physical experiment is complex. (Shape, density, initial momentum and position, ...)
- The Probability model is simple:
  - A set $\Omega$ of **outcomes**: $\Omega = \{H, T\}$. 
Random Experiment: Flip one Fair Coin

Flip a fair coin: model

- The physical experiment is complex. (Shape, density, initial momentum and position, ...)
- The Probability model is simple:
  - A set $\Omega$ of outcomes: $\Omega = \{H, T\}$.
  - A probability assigned to each outcome: $Pr[H] = 0.5, Pr[T] = 0.5$. 
Random Experiment: Flip one Unfair Coin

Flip an unfair (biased, loaded) coin:

- Possible outcomes: Heads (H) and Tails (T)
- Likelihoods:
  - H: \( p \in (0, 1) \)
  - T: \( 1 - p \)

Frequentist Interpretation:
Flip many times \( \Rightarrow \) Fraction \( 1 - p \) of tails

Question:
How can one figure out \( p \)?

Flip many times

Tautology?
No: Statistical regularity!
Random Experiment: Flip one Unfair Coin

Flip an unfair (biased, loaded) coin:
Random Experiment: Flip one Unfair Coin

Flip an unfair (biased, loaded) coin:

- Heads (H): 45%
- Tails (T): 55%
Random Experiment: Flip one Unfair Coin

Flip an **unfair** (biased, loaded) coin:

- Possible outcomes:
  - H: 45%
  - T: 55%

Frequentist Interpretation: Flip many times ⇒ Fraction $1 - p$ of tails

Question: How can one figure out $p$? Flip many times

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Flip an unfair (biased, loaded) coin:

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Random Experiment: Flip one Unfair Coin
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Flip an unfair (biased, loaded) coin: model
Random Experiment: Flip one Unfair Coin

Flip an unfair (biased, loaded) coin: model

Physical Experiment

Probability Model

$\Omega$

$H \circ p$

$T \circ 1 - p$
Flip Two Fair Coins

Possible outcomes:

\{HH, HT, TH, TT\} ≡ \{H, T\}^2.

Note:

\(A \times B := \{(a, b) | a \in A, b \in B\}\) and \(A^2 := A \times A\).

Likelihoods:

\(1/4\) each.
Flip Two Fair Coins

- Possible outcomes:
  - \{HH, HT, TH, TT\}
  - \equiv \{H, T\}^2

Note:
- \(A \times B := \{(a, b) | a \in A, b \in B\}\)
- \(A^2 := A \times A\)
- Likelihoods: 1/4 each.
Flip Two Fair Coins

- Possible outcomes: \(\{HH, HT, TH, TT\}\)
Flip Two Fair Coins

- Possible outcomes: \( \{HH, HT, TH, TT\} \equiv \{H, T\}^2 \).
Flip Two Fair Coins

- Possible outcomes: \( \{ HH, HT, TH, TT \} \equiv \{ H, T \}^2 \).
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Flip Two Fair Coins

- Possible outcomes: \( \{HH, HT, TH, TT\} \equiv \{H, T\}^2 \).
- Note: \( A \times B := \{(a, b) \mid a \in A, b \in B\} \) and \( A^2 := A \times A \).
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- Possible outcomes: \{HH, HT, TH, TT\} \equiv \{H, T\}^2.
- Note: \(A \times B := \{(a, b) \mid a \in A, b \in B\}\) and \(A^2 := A \times A\).
- Likelihoods: \(\frac{1}{4}\) each.
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- Possible outcomes: \( \{HH, HT, TH, TT\} \equiv \{H, T\}^2 \).
- Note: \( A \times B := \{(a, b) \mid a \in A, b \in B\} \) and \( A^2 := A \times A \).
- Likelihoods: \(1/4\) each.
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- Possible outcomes: \( \{HH, HT, TH, TT\} \equiv \{H, T\}^2 \).
- Note: \( A \times B := \{(a, b) \mid a \in A, b \in B\} \) and \( A^2 := A \times A \).
- Likelihoods: 1/4 each.
Flip Glued Coins

Possible outcomes:
- HH
- TT

Likelihoods:
- HH: 0.5
- TT: 0.5

Note: Coins are glued so that they show the same face.
Flip Glued Coins

Flips two coins glued together side by side:

Possible outcomes: \{HH, TT\}.
Likelihoods: HH: 0.5, TT: 0.5.

Note: Coins are glued so that they show the same face.
Flip Glued Coins

Flips two coins glued together side by side:

- Possible outcomes: {HH, TT}
- Likelihoods: HH: 0.5, TT: 0.5

Note: Coins are glued so that they show the same face.
Flip Glued Coins

Flips two coins glued together side by side:

- **Possible outcomes:**
  - HH: 0.5
  - TT: 0.5

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Flips two coins glued together side by side:

- Possible outcomes: \{HH, TT\}.

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Flips two coins glued together side by side:

- Possible outcomes: \{HH, TT\}.
- Likelihoods:

\[
\begin{align*}
HH &: 0.5, \\
TT &: 0.5.
\end{align*}
\]
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Flips two coins glued together side by side:

- Possible outcomes: \{HH, TT\}.
- Likelihoods: HH : 0.5, TT : 0.5.
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Flips two coins glued together side by side:

- Possible outcomes: \{HH, TT\}.
- Likelihoods: \(HH : 0.5, TT : 0.5\).
- Note: Coins are glued so that they show the same face.
Flip Glued Coins

Flips two coins glued together side by side:

▶ Possible outcomes: \{HT, TH\}.

▶ Likelihoods:
  - HT: 0.5
  - TH: 0.5

Note: Coins are glued so that they show different faces.
Flip Glued Coins

Flips two coins glued together side by side:
Flip Glued Coins

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Flips two coins glued together side by side:

Possible outcomes:

- HT: 50%
- TH: 50%

Note: Coins are glued so that they show different faces.
Flip Glued Coins

Flips two coins glued together side by side:

- Possible outcomes: \{HT, TH\}.

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Flip Glued Coins

Flips two coins glued together side by side:

Possible outcomes: \( \{HT, TH\} \).

Likelihoods:
Flip Glued Coins

Flips two coins glued together side by side:

- Possible outcomes: \( \{ HT, TH \} \).
- Likelihoods: \( HT : 0.5, TH : 0.5 \).
Flip Glued Coins

Flips two coins glued together side by side:

Possible outcomes: \( \{ HT, TH \} \).

Likelihoods: \( HT : 0.5, TH : 0.5 \).

Note: Coins are glued so that they show different faces.
Flip two Attached Coins

Possible outcomes: 
- HH
- HT
- TH
- TT

Likelihoods:
- HH: 0.4
- HT: 0.1
- TH: 0.1
- TT: 0.4

Note: Coins are attached so that they tend to show the same face, unless the spring twists enough.
Flip two Attached Coins

Flips two coins attached by a spring:

Possible outcomes: {HH, HT, TH, TT}.

Likelihoods: HH: 0.4, HT: 0.1, TH: 0.1, TT: 0.4.

Note: Coins are attached so that they tend to show the same face, unless the spring twists enough.
Flip two Attached Coins

Flips two coins attached by a spring:
Flip two Attached Coins

Flips two coins attached by a spring:

Possible outcomes:

- HH: 0.4
- HT: 0.1
- TH: 0.1
- TT: 0.4

Note: Coins are attached so that they tend to show the same face, unless the spring twists enough.
Flip two Attached Coins

Flips two coins attached by a spring:

- Possible outcomes: \{HH, HT, TH, TT\}.
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- Possible outcomes: \{HH, HT, TH, TT\}.
- Likelihoods:
Flip two Attached Coins

Flips two coins attached by a spring:

- Possible outcomes: \( \{HH, HT, TH, TT\} \).
- Likelihoods: \( HH : 0.4, HT : 0.1, TH : 0.1, TT : 0.4 \).
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Flipping Two Coins

Here is a way to summarize the four random experiments:

- $\Omega$ is the set of possible outcomes;
- Each outcome has a probability (likelihood);
- The probabilities are $\geq 0$ and add up to 1;
- Fair coins: [$1$];
- Glued coins: [$3$, $4$];
- Spring-attached coins: [$2$].
Here is a way to summarize the four random experiments:
Flipping Two Coins

Here is a way to summarize the four random experiments:

[1] \(\Omega\) with outcomes:
- TH: 0.25
- TT: 0.25

[2] \(\Omega\) with outcomes:
- TH: 0.1
- TT: 0.4

[3] \(\Omega\) with outcomes:
- TH: 0
- TT: 0.5

[4] \(\Omega\) with outcomes:
- TH: 0.5
- TT: 0
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Flipping Two Coins

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3. The probabilities are $\geq 0$ and add up to 1;
Flipping Two Coins

Here is a way to summarize the four random experiments:

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▶ Fair coins:
Flipping Two Coins

Here is a way to summarize the four random experiments:

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- Each outcome has a *probability* (likelihood);
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- Fair coins: [1];
Flipping Two Coins

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Flipping Two Coins

Here is a way to summarize the four random experiments:

[Diagram with outcomes and probabilities]
Flipping Two Coins

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Important remarks:
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► Each outcome describes the two coins.
Flipping Two Coins

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- E.g., $HT$ is **one** outcome of the experiment.
Flipping Two Coins

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Flipping Two Coins

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- Each $\omega \in \Omega$ describes one outcome of the complete experiment.
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▶ Indeed, this viewpoint misses the relationship between the two flips.
▶ Each $\omega \in \Omega$ describes one outcome of the complete experiment.
▶ $\Omega$ and the probabilities specify the random experiment.
Flipping $n$ times

Flip a fair coin $n$ times (some $n \geq 1$):
Flipping $n$ times

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- Possible outcomes:
Flipping $n$ times

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- Possible outcomes: \{TT \cdots T, TT \cdots H, \ldots, HH \cdots H\}.
Flipping $n$ times

Flip a fair coin $n$ times (some $n \geq 1$):

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Thus, $2^n$ possible outcomes.
Flipping $n$ times

Flip a fair coin $n$ times (some $n \geq 1$):

- Possible outcomes: $\{TT \cdots T, TT \cdots H, \ldots, HH \cdots H\}$. Thus, $2^n$ possible outcomes.

- Note: $\{TT \cdots T, TT \cdots H, \ldots, HH \cdots H\} = \{H, T\}^n$. 

\[ A^n := \{ (a_1, \ldots, a_n) \mid a_1 \in A, \ldots, a_n \in A \} \]

\[ |A^n| = |A|^n \]
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Flipping \( n \) times

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  $A^n := \{(a_1, \ldots, a_n) \mid a_1 \in A, \ldots, a_n \in A\}$. $|A^n| = |A|^n$.

- Likelihoods:
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  $A^n := \{(a_1, \ldots, a_n) \mid a_1 \in A, \ldots, a_n \in A\}$. $|A^n| = |A|^n$.
- Likelihoods: $1/2^n$ each.
Flipping $n$ times

Flip a fair coin $n$ times (some $n \geq 1$):

▶ Possible outcomes: $\{TT \cdots T, TT \cdots H, \ldots, HH \cdots H\}$.

Thus, $2^n$ possible outcomes.

▶ Note: $\{TT \cdots T, TT \cdots H, \ldots, HH \cdots H\} = \{H, T\}^n$.

$A^n := \{(a_1, \ldots, a_n) \mid a_1 \in A, \ldots, a_n \in A\}$. $|A^n| = |A|^n$.

▶ Likelihoods: $1/2^n$ each.
Roll two Dice

Roll a *balanced* 6-sided die twice:
Roll two Dice

Roll a balanced 6-sided die twice:

- Possible outcomes:
Roll two Dice

Roll a balanced 6-sided die twice:

- Possible outcomes:
  \[ \{1, 2, 3, 4, 5, 6\}^2 = \{(a, b) \mid 1 \leq a, b \leq 6\}. \]
Roll two Dice

Roll a balanced 6-sided die twice:

- Possible outcomes:
  \[\{1, 2, 3, 4, 5, 6\}^2 = \{(a, b) \mid 1 \leq a, b \leq 6\}\.\]
- Likelihoods:
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- Possible outcomes:
  \[ \{1, 2, 3, 4, 5, 6\}^2 = \{(a, b) \mid 1 \leq a, b \leq 6\}. \]
- Likelihoods: 1/36 for each.
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Roll a **balanced** 6-sided die twice:

- Possible outcomes:
  \[ \{1, 2, 3, 4, 5, 6\}^2 = \{(a, b) \mid 1 \leq a, b \leq 6\} \]

- Likelihoods: 1/36 for each.

![Physical Experiment](image1.png)  ![Probability Model](image2.png)
Probability Space.

1. A “random experiment”:

(a) Flip a biased coin;
(b) Flip two fair coins;
(c) Deal a poker hand.

2. A set of possible outcomes: \( \Omega \).

(a) \( \Omega = \{H, T\} \);
(b) \( \Omega = \{HH, HT, TH, TT\} \); \( |\Omega| = 4 \);
(c) \( \Omega = \{A\spadesuit A\heartsuit A\clubsuit A\diamondsuit K\spadesuit, \ldots\} \); \( |\Omega| = \binom{52}{5} \).

3. Assign a probability to each outcome: \( \Pr : \Omega \rightarrow [0, 1] \).

(a) \( \Pr[H] = p, \Pr[T] = 1 - p \) for some \( p \in [0, 1] \);
(b) \( \Pr[HH] = \Pr[HT] = \Pr[TH] = \Pr[TT] = 1/4 \);
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Probability Space.

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Probability Space.

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Probability Space: formalism.

Ω is the **sample space**.
Probability Space: formalism.

Ω is the **sample space**.
ω ∈ Ω is a **sample point**.
Probability Space: formalism.

\( \Omega \) is the \textbf{sample space}.
\( \omega \in \Omega \) is a \textbf{sample point}. (Also called an \textbf{outcome}.)
Probability Space: formalism.

$\Omega$ is the **sample space**.
$\omega \in \Omega$ is a **sample point**. (Also called an **outcome**.)
Sample point $\omega$ has a probability $Pr[\omega]$ where
Probability Space: formalism.

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Probability Space: formalism.

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- \( 0 \leq Pr[\omega] \leq 1 \);
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Probability Space: formalism.

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Sample point $\omega$ has a probability $Pr[\omega]$ where

- $0 \leq Pr[\omega] \leq 1$;
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![Sample Space Diagram](image)
In a **uniform probability space** each outcome $\omega$ is **equally probable**: $Pr[\omega] = \frac{1}{|\Omega|}$ for all $\omega \in \Omega$. 

Examples:

- Flipping two fair coins, dealing a poker hand are uniform probability spaces.
- Flipping a biased coin is not a uniform probability space.
In a **uniform probability space** each outcome $\omega$ is **equally probable**: $Pr[\omega] = \frac{1}{|\Omega|}$ for all $\omega \in \Omega$. 

![Uniform Probability Space Diagram](image)
In a **uniform probability space** each outcome \( \omega \) is equally probable: \( Pr[\omega] = \frac{1}{|\Omega|} \) for all \( \omega \in \Omega \).

Examples:
- Flipping two fair coins, dealing a poker hand are uniform probability spaces.
Probability Space: Formalism.

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Examples:

- Flipping two fair coins, dealing a poker hand are uniform probability spaces.
- Flipping a biased coin is not a uniform probability space.
Probability Space: Formalism

Simplest physical model of a uniform probability space:
Probability Space: Formalism

Simplest physical model of a **uniform** probability space:

- **Physical experiment**
  - A bag of identical balls, except for their color (or a label).
  - If the bag is well shaken, every ball is equally likely to be picked.
  - $\Omega = \{\text{white, red, yellow, grey, purple, blue, maroon, green}\}$

- **Probability model**
  - $\Omega$
  - $Pr[\omega]$
  - - Red: $1/8$
  - - Green: $1/8$
  - - Maroon: $1/8$

A bag of identical balls, except for their color (or a label).

\[
\Omega = \{\text{white, red, yellow, grey, purple, blue, maroon, green}\}
\]

\[
Pr[\omega] = \frac{1}{8}
\]

\[
\begin{aligned}
\bullet \text{ Red} & : \frac{1}{8} \\
\bullet \text{ Green} & : \frac{1}{8} \\
\vdots & \\
\bullet \text{ Maroon} & : \frac{1}{8}
\end{aligned}
\]
Probability Space: Formalism

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A bag of identical balls, except for their color (or a label). If the bag is well shaken, every ball is equally likely to be picked.
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Probability Space: Formalism

Simplest physical model of a uniform probability space:

A bag of identical balls, except for their color (or a label). If the bag is well shaken, every ball is equally likely to be picked.

\[ \Omega = \{ \text{white, red, yellow, grey, purple, blue, maroon, green} \} \]

\[ Pr[\text{blue}] = 1/8 \]
A bag of identical balls, except for their color (or a label). If the bag is well shaken, every ball is equally likely to be picked.

\[ \Omega = \{ \text{white, red, yellow, grey, purple, blue, maroon, green} \} \]

\[ Pr[\text{blue}] = \frac{1}{8}. \]
Probability Space: Formalism

Simplest physical model of a non-uniform probability space:

$\Omega = \{\text{Red, Green, Yellow, Blue}\}$

$\Pr[\text{Red}] = \frac{3}{10}$,

$\Pr[\text{Green}] = \frac{4}{10}$,

etc.

Note: Probabilities are restricted to rational numbers: $\frac{p}{q}$.
Probability Space: Formalism

Simplest physical model of a non-uniform probability space:

\[ \Omega = \{ \text{Red, Green, Yellow, Blue} \} \]

\[ Pr[\text{Red}] = \frac{3}{10}, \quad Pr[\text{Green}] = \frac{4}{10}, \quad \text{etc.} \]

Note: Probabilities are restricted to rational numbers: \( \mathbb{Q} \).

Physical experiment

Probability model
Probability Space: Formalism

Simplest physical model of a non-uniform probability space:

\[ \Omega = \{ \text{Red, Green, Yellow, Blue} \} \]
Probability Space: Formalism

Simplest physical model of a non-uniform probability space:

\[ \Omega = \{ \text{Red, Green, Yellow, Blue} \} \]

\[ Pr[\text{Red}] = \]
Probability Space: Formalism

Simplest physical model of a non-uniform probability space:

\[ \Omega = \{ \text{Red, Green, Yellow, Blue} \} \]

\[ Pr[\text{Red}] = \frac{3}{10}, \]

\[ Pr[\text{Green}] = \frac{4}{10}, \]

\[ Pr[\text{Yellow}] = \frac{2}{10}, \]

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Simplest physical model of a non-uniform probability space:

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Probability Space: Formalism

Simplest physical model of a non-uniform probability space:

\[ \Omega = \{ \text{Red, Green, Yellow, Blue} \} \]

\[ Pr[\text{Red}] = \frac{3}{10}, Pr[\text{Green}] = \frac{4}{10}, \text{etc.} \]

Note: Probabilities are restricted to rational numbers: \( \frac{N_k}{N} \).
Probability Space: Formalism

Physical model of a general non-uniform probability space:
Probability Space: Formalism

Physical model of a general non-uniform probability space:

\( \Omega = \{1, 2, 3, \ldots, N\} \),

\( \Pr[\omega] = p_\omega \).

Physical experiment

Probability model

- Green = 1
- Purple = 2
- Yellow = \( \omega \)

\( p_1, p_2, \ldots, p_\omega \)

\( p_1, p_2, \ldots, p_\omega \)

The roulette wheel stops in sector \( \omega \) with probability \( p_\omega \).
Probability Space: Formalism

Physical model of a general non-uniform probability space:

The roulette wheel stops in sector $\omega$ with probability $p_\omega$.  

\[ \Omega = \{1, 2, 3, \ldots, N\} \]

\[ Pr[\omega] = p_\omega \]
Probability Space: Formalism

Physical model of a general non-uniform probability space:

The roulette wheel stops in sector $\omega$ with probability $p_\omega$.

$$\Omega = \{1, 2, 3, \ldots, N\},$$
Probability Space: Formalism

Physical model of a general non-uniform probability space:

The roulette wheel stops in sector $\omega$ with probability $p_\omega$.

$$\Omega = \{1, 2, 3, \ldots, N\}, \Pr[\omega] = p_\omega.$$
An important remark

- The random experiment selects **one and only one** outcome in $\Omega$. 
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An important remark

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  - The experiment selects **one** of the elements of $\Omega$.
- In this case, it would be wrong to think that $\Omega = \{H, T\}$ and that the experiment selects two outcomes.
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- Why?
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- For instance, when we flip a fair coin twice
  - $\Omega = \{HH, TH, HT, TT\}$
  - The experiment selects *one* of the elements of $\Omega$.
- In this case, it's would be wrong to think that $\Omega = \{H, T\}$ and that the experiment selects two outcomes.
- Why? Because this would not describe how the two coin flips are related to each other.
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- For instance, say we glue the coins side-by-side so that they face up the same way.
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- Why? Because this would not describe how the two coin flips are related to each other.
- For instance, say we glue the coins side-by-side so that they face up the same way. Then one gets $HH$ or $TT$ with probability 50% each.
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- For instance, when we flip a fair coin **twice**
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- In this case, it’s would be wrong to think that $\Omega = \{H, T\}$ and that the experiment selects two outcomes.
- Why? Because this would not describe how the two coin flips are related to each other.
- For instance, say we glue the coins side-by-side so that they face up the same way. Then one gets $HH$ or $TT$ with probability 50% each. This is not captured by ‘picking two outcomes.’
Lecture 15: Summary

Modeling Uncertainty: Probability Space

1. Random Experiment
2. Probability Space: \( \Omega; \Pr(\omega) \in [0, 1]; \sum_{\omega} \Pr(\omega) = 1. \)
3. Uniform Probability Space: \( \Pr(\omega) = \frac{1}{|\Omega|} \) for all \( \omega \in \Omega \).
Lecture 15: Summary

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