Today: Proofs!!

1. By Example.
2. Direct. (Prove $P \implies Q$. )
3. by Contraposition (Prove $P \implies Q$)
4. by Contradiction (Prove $P$.)
5. by Cases
Quick Background and Notation.

Integers closed under addition.
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\[ a, b \in \mathbb{Z} \implies a + b \in \mathbb{Z} \]
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\( a \mid b \) means “a divides b”.

2 \mid 4? Yes!

7 \mid 23? No!

4 \mid 2? No!

Formally:

\[ a \mid b \iff \exists q \in \mathbb{Z} \text{ where } b = aq \]

3 \mid 15 since for \( q = 5 \), 15 = 3(5).

A natural number \( p > 1 \), is prime if it is divisible only by 1 and itself.
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A natural number \( p > 1 \), is **prime** if it is divisible only by 1 and itself.
Direct Proof.

**Theorem:** For any $a, b, c \in \mathbb{Z}$, if $a|b$ and $a|c$ then $a|(b - c)$.
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Argument applies to every $a, b, c \in \mathbb{Z}$. 

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**Direct Proof Form:**

Goal: \(P \implies Q\)
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\[ (b - c) = a(q - q') \quad \text{and} \quad (q - q') \quad \text{is an integer so} \]

$\quad a | (b - c)$  \hspace{1cm} \square$

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Direct Proof Form:

Goal: $P \implies Q$

Assume $P$.

\[ \ldots \]

Therefore $Q$. 

Another direct proof.

Let $D_3$ be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, than $11 \mid n$.

Examples:

$n = 121$

Alt. Sum: $1 - 2 + 1 = 0$. Divis. by 11.

As is 121.

$n = 605$


As is 605 = 11 (55).

Proof:

For $n \in D_3$, $n = 100a + 10b + c$, for some $a, b, c$.

Assume: Alt. sum: $a - b + c = 11k$ for some integer $k$.

Add 99$a + 11b$ to both sides.

$100a + 10b + c = 11k + 99a + 11b = 11(k + 9a + b)$

Left hand side is $n$, $k + 9a + b$ is integer.

$\Rightarrow 11 \mid n$.

Direct proof of $P \Rightarrow Q$:

Assumed $P$: $11 \mid a - b + c$.

Proved $Q$: $11 \mid n$. 
Another direct proof.

Let $D_3$ be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, then $11|n$.

Examples:

- $n = 121$
  - Alt Sum: $1 - 2 + 1 = 0$.
  - Divisible by 11.
  - As is $121 = 11 \times 11$

Proof:

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Assume: Alt. sum: $a - b + c = 11k$ for some integer $k$.

Add $99a + 11b$ to both sides.

$100a + 10b + c = 11k + 99a + 11b$

The left hand side is $n$, $k + 9a + b$ is integer.

$\implies 11|n$. 

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Alt Sum: $6 - 0 + 5 = 11$.

Divis. by 11.

As is $605 = 11 \cdot 55$.

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Let $D_3$ be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, than $11 \mid n$.

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\forall n \in D_3, (11 \mid \text{alt. sum of digits of } n) \implies 11 \mid n
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n = 121  Alt Sum: $1 - 2 + 1 = 0$. Divis. by 11. As is 121.
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Proof: For $n \in D_3$, $n = 100a + 10b + c$, for some $a, b, c$. 
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Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, than $11|n$.

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Let \( D_3 \) be the 3 digit natural numbers.

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**Proof:** For \( n \in D_3 \), \( n = 100a + 10b + c \), for some \( a, b, c \).

Assume: Alt. sum: \( a - b + c = 11k \) for some integer \( k \).

Add \( 99a + 11b \) to both sides.
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$$100a + 10b + c = 11k + 99a + 11b$$
Another direct proof.

Let $D_3$ be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, than $11|n$.

$$\forall n \in D_3, (11|\text{alt. sum of digits of } n) \implies 11|n$$

Examples:
$n = 121$  Alt Sum: $1 - 2 + 1 = 0$. Divis. by 11. As is 121.
$n = 605$  Alt Sum: $6 - 0 + 5 = 11$ Divis. by 11. As is $605 = 11(55)$

Proof: For $n \in D_3$, $n = 100a + 10b + c$, for some $a, b, c$.

Assume: Alt. sum: $a - b + c = 11k$ for some integer $k$.

Add $99a + 11b$ to both sides.

$$100a + 10b + c = 11k + 99a + 11b = 11(k + 9a + b)$$
Another direct proof.

Let \( D_3 \) be the 3 digit natural numbers.

Theorem: For \( n \in D_3 \), if the alternating sum of digits of \( n \) is divisible by 11, then \( 11 \mid n \).

\[
\forall n \in D_3, (11 \mid \text{alt. sum of digits of } n) \implies 11 \mid n
\]

Examples:

\( n = 121 \)  \( \text{Alt Sum: } 1 - 2 + 1 = 0 \). Divis. by 11. As is 121.

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Add \( 99a + 11b \) to both sides.

\[
100a + 10b + c = 11k + 99a + 11b = 11(k + 9a + b)
\]

Left hand side is \( n \),
Another direct proof.

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Left hand side is $n$, $k + 9a + b$ is integer.
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Left hand side is $n$, $k + 9a + b$ is integer.  $\implies 11 \mid n$.  \(\square\)

Direct proof of $P \implies Q$:

Assumed $P$: $11 \mid a - b + c$.  

Another direct proof.

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Left hand side is $n$, $k + 9a + b$ is integer.  \( \implies 11|n. \) \[ \square \]

Direct proof of $P \implies Q$:
Assumed $P$: $11|a - b + c$. Proved $Q$: $11|n$.  \[ \square \]
The Converse

Thm: \( \forall n \in D_3, (11 \mid \text{alt. sum of digits of } n) \implies 11 \mid n \)
The Converse

Thm: $\forall n \in D_3, (11 | \text{alt. sum of digits of } n) \implies 11 | n$

Is converse a theorem?
$\forall n \in D_3, (11 | n) \implies (11 | \text{alt. sum of digits of } n)$
The Converse

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$\forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n)$

Yes?
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Is converse a theorem?
\( \forall n \in D_3, (11 \mid n) \implies (11 \mid \text{alt. sum of digits of } n) \)

Yes? No?
Another Direct Proof.

Theorem: \( \forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n) \)
Another Direct Proof.

Theorem: \( \forall n \in D_3, (11 | n) \implies (11 | \text{alt. sum of digits of } n) \)

Proof:
Another Direct Proof.

Theorem: \( \forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n) \)
Proof: Assume 11|n.
Theorem: \( \forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n) \)

Proof: Assume \( 11|n \).

\[ n = 100a + 10b + c = 11k \]
Another Direct Proof.

Theorem: $\forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n)$

Proof: Assume $11|n$.

$$n = 100a + 10b + c = 11k \implies 99a + 11b + (a - b + c) = 11k$$
Another Direct Proof.

Theorem: \( \forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n) \)

Proof: Assume \( 11|n \).

\[
n = 100a + 10b + c = 11k \implies 99a + 11b + (a - b + c) = 11k \implies a - b + c = 11k - 99a - 11b
\]
Another Direct Proof.

Theorem: \( \forall n \in D_3, (11 | n) \implies (11 | \text{alt. sum of digits of } n) \)

Proof: Assume \( 11 | n \).

\[
\begin{align*}
n &= 100a + 10b + c = 11k \\ 99a + 11b + (a - b + c) &= 11k \\ a - b + c &= 11k - 99a - 11b \\ a - b + c &= 11(k - 9a - b)
\end{align*}
\]
Theorem: \( \forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n) \)

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\[ n = 100a + 10b + c = 11k \implies \]
\[ 99a + 11b + (a - b + c) = 11k \implies \]
\[ a - b + c = 11k - 99a - 11b \implies \]
\[ a - b + c = 11(k - 9a - b) \implies \]
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That is \( 11|\text{alternating sum of digits of } n \).  \(\Box\)
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That is $11|\text{alternating sum of digits}$. \qed

Note: similar proof to other. In this case every $\implies$ is $\iff$. 
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a - b + c &= 11\ell \text{ where } \ell = (k - 9a - b) \in Z
\end{align*}
\]

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Note: similar proof to other. In this case every \( \implies \) is \( \iff \).

Often works with arithmetic properties ...
Another Direct Proof.

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\]

That is \( 11|\text{alternating sum of digits} \).

Note: similar proof to other. In this case every \( \implies \) is \( \iff \)

Often works with arithmetic properties ...
...not when multiplying by 0.
Another Direct Proof.

Theorem: \( \forall n \in D_3, (11|n) \iff (11|\text{alt. sum of digits of } n) \)

Proof: Assume 11\( |n \).

\[
n = 100a + 10b + c = 11k \implies
99a + 11b + (a - b + c) = 11k \implies
a - b + c = 11k - 99a - 11b \implies
a - b + c = 11(k - 9a - b) \implies
a - b + c = 11\ell \quad \text{where } \ell = (k - 9a - b) \in \mathbb{Z}
\]

That is 11\( |\text{alternating sum of digits} \).

Note: similar proof to other. In this case every \( \implies \) is \( \iff \)

Often works with arithmetic properties ...

...not when multiplying by 0.

We have.
Another Direct Proof.

Theorem: $\forall n \in D_3, (11 \mid n) \implies (11 \mid \text{alt. sum of digits of } n)$

Proof: Assume $11 \mid n$.

\[
n = 100a + 10b + c = 11k \implies 99a + 11b + (a - b + c) = 11k \implies a - b + c = 11(k - 9a - b) \implies a - b + c = 11\ell \text{ where } \ell = (k - 9a - b) \in \mathbb{Z}
\]

That is $11 \mid \text{alternating sum of digits}$.

Note: similar proof to other. In this case every $\implies$ is $\iff$

Often works with arithmetic properties ...

...not when multiplying by 0.

We have.

Theorem: $\forall n \in N', (11 \mid \text{alt. sum of digits of } n) \iff (11 \mid n)$
Proof by Contraposition

Thm: For $n \in \mathbb{Z}^+$ and $d \mid n$. If $n$ is odd then $d$ is odd.

$n = 2^k + 1$

What do we know about $d$?

Goal: Prove $P \Rightarrow Q$.

Assume $\neg Q$ ... and prove $\neg P$.

Conclusion: $\neg Q = \Rightarrow \neg P$.

Equivalent to $P \Rightarrow Q$.

Proof:

Assume $\neg Q$: $d$ is even.

$d = 2^k$.

$d \mid n$ so we have $n = qd = q(2^k) = 2^{k+1}q$.

$n$ is even. $\neg P$
Proof by Contraposition

Thm: For \( n \in \mathbb{Z}^+ \) and \( d \mid n \). If \( n \) is odd then \( d \) is odd.

\[ n = 2k + 1 \]

What do we know about \( d \)?

What to do?

Goal: Prove \( P \Rightarrow Q \).

Assume \( \neg Q \) ...

and prove \( \neg P \).

Conclusion:

\( \neg Q \Rightarrow \neg P \).

Proof:

Assume \( \neg Q \): \( d \) is even.

\[ d = 2k \]

\( d \mid n \) so we have

\[ n = qd = q(2k) = 2(kq) \]

\( n \) is even.

\( \neg P \)
Proof by Contraposition

Thm: For \( n \in Z^+ \) and \( d \mid n \). If \( n \) is odd then \( d \) is odd.

\( n = 2k + 1 \)
Proof by Contraposition

Thm: For $n \in Z^+$ and $d|n$. If $n$ is odd then $d$ is odd.

$n = 2k + 1$ what do we know about $d$?
Proof by Contraposition

Thm: For $n \in \mathbb{Z}^+$ and $d|n$. If $n$ is odd then $d$ is odd.

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What to do?
Proof by Contraposition

Thm: For $n \in \mathbb{Z}^+$ and $d \mid n$. If $n$ is odd then $d$ is odd.

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Goal: Prove $P \implies Q$. 

Proof by Contraposition

Thm: For $n \in \mathbb{Z}^+$ and $d \mid n$. If $n$ is odd then $d$ is odd.

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What to do?

Goal: Prove $P \implies Q$. 

$\square$
Thm: For $n \in \mathbb{Z}^+$ and $d|n$. If $n$ is odd then $d$ is odd.

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What to do?

Goal: Prove $P \implies Q$.

Assume $\neg Q$
Proof by Contraposition

Thm: For $n \in \mathbb{Z}^+$ and $d | n$. If $n$ is odd then $d$ is odd.

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What to do?

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Assume $\neg Q$

...and prove $\neg P$.

Conclusion: $\neg Q \implies \neg P$
Thm: For $n \in \mathbb{Z}^+$ and $d | n$. If $n$ is odd then $d$ is odd.

$n = 2k + 1$ what do we know about $d$?

What to do?

Goal: Prove $P \implies Q$.

Assume $\neg Q$

...and prove $\neg P$.

Conclusion: $\neg Q \implies \neg P$ equivalent to $P \implies Q$. 

Proof:

Assume $\neg Q$: $d$ is even.

$d = 2k$.

$d | n$ so we have $n = qd = q(2k)$.

$n$ is even.

$\neg P$
Thm: For $n \in \mathbb{Z}^+$ and $d | n$. If $n$ is odd then $d$ is odd.

$n = 2k + 1$ what do we know about $d$?

What to do?

Goal: Prove $P \implies Q$.

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...and prove $\neg P$.

Conclusion: $\neg Q \implies \neg P$ equivalent to $P \implies Q$.

Proof: Assume $\neg Q$: $d$ is even.
Proof by Contraposition

Thm: For $n \in \mathbb{Z}^+$ and $d | n$. If $n$ is odd then $d$ is odd.

$n = 2k + 1$ what do we know about $d$?

What to do?

Goal: Prove $P \implies Q$.

Assume $\neg Q$

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Proof: Assume $\neg Q$: $d$ is even. $d = 2k$. 
Proof by Contraposition

Thm: For \( n \in \mathbb{Z}^+ \) and \( d \mid n \). If \( n \) is odd then \( d \) is odd.

\[ n = 2k + 1 \] what do we know about \( d \)?

What to do?

Goal: Prove \( P \implies Q \).

Assume \( \neg Q \)

...and prove \( \neg P \).

Conclusion: \( \neg Q \implies \neg P \) equivalent to \( P \implies Q \).

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\( d \mid n \) so we have
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Goal: Prove \( P \implies Q \).

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\( d|n \) so we have

\( n = qd \)
Proof by Contraposition

Thm: For \( n \in \mathbb{Z}^+ \) and \( d \mid n \). If \( n \) is odd then \( d \) is odd.

\[ n = 2k + 1 \] what do we know about \( d \)?

What to do?

Goal: Prove \( P \implies Q \).

Assume \( \neg Q \)

...and prove \( \neg P \).

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Proof: Assume \( \neg Q \): \( d \) is even. \( d = 2k \).

\( d \mid n \) so we have

\[ n = qd = q(2k) \]
Proof by Contraposition

Thm: For $n \in \mathbb{Z}^+$ and $d | n$. If $n$ is odd then $d$ is odd.

$n = 2k + 1$ what do we know about $d$?

What to do?

Goal: Prove $P \implies Q$.

Assume $\neg Q$

...and prove $\neg P$.

Conclusion: $\neg Q \implies \neg P$ equivalent to $P \implies Q$.

Proof: Assume $\neg Q$: $d$ is even. $d = 2k$.

$d | n$ so we have

$n = qd = q(2k) = 2(kq)$
Proof by Contraposition

Thm: For $n \in \mathbb{Z}^+$ and $d|n$. If $n$ is odd then $d$ is odd.

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$d|n$ so we have

$n = qd = q(2k) = 2(kq)$

$n$ is even. $\neg P$
Another Contraposition...

Lemma: For every \( n \) in \( \mathbb{N} \), \( n^2 \) is even \( \Rightarrow \) \( n \) is even. (\( P \Rightarrow Q \))

Proof by contraposition: (\( P \Rightarrow Q \)) \( \equiv \) (\( \neg Q \Rightarrow \neg P \))

\( P \) = \' \( n^2 \) is even.\' ...........

\( \neg P \) = \' \( n^2 \) is odd\' ...........

\( Q \) = \' \( n \) is even\' ...........

\( \neg Q \) = \' \( n \) is odd\'

Prove \( \neg Q \Rightarrow \neg P \): \( n \) is odd \( \Rightarrow \) \( n^2 \) is odd.

\( n = 2k + 1 \)

\( n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 \).

\( n^2 = 2l + 1 \) where \( l \) is a natural number.

... and \( n^2 \) is odd!

\( \neg Q \Rightarrow \neg P \) so \( P \Rightarrow Q \) and ...
Another Contraposition...

**Lemma:** For every $n$ in $N$, $n^2$ is even $\iff$ $n$ is even. ($P \iff Q$)

Proof by contraposition: ($P \iff Q$) $\equiv$ ($\neg Q \iff \neg P$)

$P = 'n^2$ is even.'

$\neg P = 'n^2$ is odd.'

$Q = 'n$ is even.'

$\neg Q = 'n$ is odd.'

Prove $\neg Q \iff \neg P$:

$n$ is odd $\iff n^2$ is odd.

$n = 2k + 1$

$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.

$n^2 = 2l + 1$ where $l$ is a natural number.

... and $n^2$ is odd!

$\neg Q \iff \neg P$ so $P \iff Q$ and ...
Another Contraposition...

**Lemma:** For every $n$ in $N$, $n^2$ is even $\implies n$ is even. ($P \implies Q$)

$n^2$ is even, $n^2 = 2k$, ...
Another Contraposition...

**Lemma:** For every $n$ in $N$, $n^2$ is even $\implies$ $n$ is even. ($P \implies Q$)

$n^2$ is even, $n^2 = 2k$, ... $\sqrt{2k}$ even?
Another Contraposition...

**Lemma:** For every $n$ in $N$, $n^2$ is even $\iff n$ is even. ($P \implies Q$)

**Proof by contraposition:** ($P \implies Q$) $\equiv$ ($\neg Q \implies \neg P$)
Lemma: For every $n$ in $\mathbb{N}$, $n^2$ is even $\implies n$ is even. ($P \implies Q$)

Proof by contraposition: $(P \implies Q) \equiv (\neg Q \implies \neg P)$

$P = \text{"n}^2\text{" is even.}"$ ............

$n^2 = 2k + 1$ where $k$ is a natural number.
Lemma: For every $n$ in $\mathbb{N}$, $n^2$ is even $\iff n$ is even. ($P \iff Q$)

Proof by contraposition: ($P \iff Q) \equiv (\neg Q \iff \neg P$)

$P = 'n^2$ is even.' ........... $\neg P = 'n^2$ is odd'
Another Contraposition...

**Lemma:** For every $n$ in $N$, $n^2$ is even $\implies n$ is even. ($P \implies Q$)

**Proof by contraposition:** ($P \implies Q) \equiv (\neg Q \implies \neg P$)

$P = 'n^2$ is even.' ........... $\neg P = 'n^2$ is odd'$

$Q = 'n$ is even' ..........
Lemma: For every $n$ in $N$, $n^2$ is even $\iff$ $n$ is even. ($P \iff Q$)

Proof by contraposition: ($P \iff Q) \equiv (\neg Q \iff \neg P$)

$P = 'n^2$ is even.' ........... $\neg P = 'n^2$ is odd'

$Q = 'n$ is even' ........... $\neg Q = 'n$ is odd'
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Proof by contraposition: \((P \implies Q) \equiv (\neg Q \implies \neg P)\)

\( P = \) ‘\( n^2 \) is even.’ ........... \( \neg P = \) ‘\( n^2 \) is odd’

\( Q = \) ‘\( n \) is even’ ........... \( \neg Q = \) ‘\( n \) is odd’

Prove \( \neg Q \implies \neg P \): \( n \) is odd \( \implies n^2 \) is odd.
Lemma: For every $n$ in $N$, $n^2$ is even $\implies n$ is even. ($P \implies Q$)

Proof by contraposition: ($P \implies Q \equiv (\neg Q \implies \neg P)$

$P = 'n^2$ is even.' ........ $\neg P = 'n^2$ is odd' 

$Q = 'n$ is even' ........ $\neg Q = 'n$ is odd' 

Prove $\neg Q \implies \neg P$: $n$ is odd $\implies n^2$ is odd. 
$n = 2k + 1$
Lemma: For every $n$ in $N$, $n^2$ is even $\implies$ $n$ is even. ($P \implies Q$)

Proof by contraposition: ($P \implies Q$) $\equiv$ ($\neg Q \implies \neg P$)

$P = 'n^2$ is even.' ........... $\neg P = 'n^2$ is odd'

$Q = 'n$ is even' ........... $\neg Q = 'n$ is odd'

Prove $\neg Q \implies \neg P$: $n$ is odd $\implies n^2$ is odd.

$n = 2k + 1$

$n^2 = 4k^2 + 4k + 1 = 2(2k + k) + 1$. 
Lemma: For every $n$ in $N$, $n^2$ is even $\Rightarrow$ $n$ is even. $(P \Rightarrow Q)$

Proof by contraposition: $(P \Rightarrow Q) \equiv (\neg Q \Rightarrow \neg P)$

$P = 'n^2$ is even.' ........... $\neg P = 'n^2$ is odd'$

$Q = 'n$ is even' ........... $\neg Q = 'n$ is odd'$

Prove $\neg Q \Rightarrow \neg P$: $n$ is odd $\Rightarrow$ $n^2$ is odd.

$n = 2k + 1$

$n^2 = 4k^2 + 4k + 1 = 2(2k + k) + 1$.

$n^2 = 2l + 1$ where $l$ is a natural number..
Another Contraposition...

**Lemma:** For every $n$ in $\mathbb{N}$, $n^2$ is even $\implies n$ is even. ($P \implies Q$)

**Proof by contraposition:** $(P \implies Q) \equiv (\neg Q \implies \neg P)$

$P = \text{‘}n^2 \text{ is even.’ ........... } \neg P = \text{‘}n^2 \text{ is odd’}$

$Q = \text{‘}n \text{ is even’ ........... } \neg Q = \text{‘}n \text{ is odd’}$

Prove $\neg Q \implies \neg P$: $n$ is odd $\implies n^2$ is odd.

$n = 2k + 1$

$n^2 = 4k^2 + 4k + 1 = 2(2k + k) + 1$.

$n^2 = 2l + 1$ where $l$ is a natural number..

... and $n^2$ is odd!
Lemma: For every \( n \) in \( \mathbb{N} \), \( n^2 \) is even \( \implies \) \( n \) is even. \( (P \implies Q) \)

Proof by contraposition: \( (P \implies Q) \equiv (\neg Q \implies \neg P) \)

\( P = \) ’\( n^2 \) is even.’ ........... \( \neg P = \) ’\( n^2 \) is odd’

\( Q = \) ’\( n \) is even’ ........... \( \neg Q = \) ’\( n \) is odd’

Prove \( \neg Q \implies \neg P \): \( n \) is odd \( \implies \) \( n^2 \) is odd.

\( n = 2k + 1 \)

\( n^2 = 4k^2 + 4k + 1 = 2(2k + k) + 1. \)

\( n^2 = 2l + 1 \) where \( l \) is a natural number..

... and \( n^2 \) is odd!

\( \neg Q \implies \neg P \)
Another Contraposition...

**Lemma:** For every $n$ in $N$, $n^2$ is even $\implies$ $n$ is even. $(P \implies Q)$

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$P = 'n^2$ is even.' .............. $\neg P = 'n^2$ is odd'$

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Prove $\neg Q \implies \neg P$: $n$ is odd $\implies n^2$ is odd.

$n = 2k + 1$

$n^2 = 4k^2 + 4k + 1 = 2(2k + k) + 1.$

$n^2 = 2l + 1$ where $l$ is a natural number..

... and $n^2$ is odd!

$\neg Q \implies \neg P$ so $P \implies Q$ and ...
Lemma: For every $n$ in $N$, $n^2$ is even $\implies n$ is even. ($P \implies Q$)

Proof by contraposition: ($P \implies Q) \equiv (\neg Q \implies \neg P$)

$P = 'n^2$ is even.' ............ $\neg P = 'n^2$ is odd'

$Q = 'n$ is even' ............ $\neg Q = 'n$ is odd'

Prove $\neg Q \implies \neg P$: $n$ is odd $\implies n^2$ is odd.

$n = 2k + 1$

$n^2 = 4k^2 + 4k + 1 = 2(2k + k) + 1$

$n^2 = 2l + 1$ where $l$ is a natural number..

... and $n^2$ is odd!

$\neg Q \implies \neg P$ so $P \implies Q$ and ...
Proof by contradiction: form

**Theorem:** $\sqrt{2}$ is irrational.
Proof by contradiction: form

**Theorem:** $\sqrt{2}$ is irrational.

Must show:
Proof by contradiction: form

**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$,
Proof by contradiction: form

**Theorem:** \( \sqrt{2} \) is irrational.

Must show: For every \( a, b \in \mathbb{Z} \), \((\frac{a}{b})^2 \neq 2\).
**Theorem:** \( \sqrt{2} \) is irrational.

Must show: For every \( a, b \in \mathbb{Z} \), \( \left( \frac{a}{b} \right)^2 \neq 2 \).

A simple property (equality) should always “not” hold.
Proof by contradiction: form

**Theorem:** \( \sqrt{2} \) is irrational.

Must show: For every \( a, b \in \mathbb{Z} \), \( \left( \frac{a}{b} \right)^2 \neq 2 \).

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Proof by contradiction:
Theorem: \( \sqrt{2} \) is irrational.

Must show: For every \( a, b \in \mathbb{Z} \), \( (\frac{a}{b})^2 \neq 2 \).

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Proof by contradiction:

Theorem: \( P \).
**Theorem**: $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, $\left(\frac{a}{b}\right)^2 \neq 2$.

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem**: $P$.

$\neg P$
Proof by contradiction: form

**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, $(\frac{a}{b})^2 \neq 2$.

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem:** $P$.

$\neg P \implies P_1$
Proof by contradiction: form

**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, $\left(\frac{a}{b}\right)^2 \neq 2$.

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem:** $P$.

$\neg P \implies P_1 \ldots$
Proof by contradiction: form

**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, $(\frac{a}{b})^2 \neq 2$.

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem:** $P$.

$\neg P \implies P_1 \ldots \implies R$
Proof by contradiction: form

**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, $(\frac{a}{b})^2 \neq 2$.

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem:** $P$.

$\neg P \implies P_1 \cdots \implies R$

$\neg P$
**Theorem:** \( \sqrt{2} \) is irrational.

Must show: For every \( a, b \in \mathbb{Z} \), \( \left( \frac{a}{b} \right)^2 \neq 2 \).

A simple property (equality) should always "not" hold.

Proof by contradiction:

**Theorem:** \( P \).

\[ \neg P \implies P_1 \cdots \implies R \]

\[ \neg P \implies Q_1 \]
Proof by contradiction: form

**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, $(\frac{a}{b})^2 \neq 2$.

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**Theorem:** $P$.

$\neg P \implies P_1 \cdots \implies R$

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$\neg P \implies P_1 \cdots \implies R$

$\neg P \implies Q_1 \cdots \implies \neg R$

$\neg P \implies R \land \neg R$
**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, $(\frac{a}{b})^2 \neq 2$.

A simple property (equality) should always “not” hold.

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**Theorem:** $P$.

$\neg P \implies P_1 \cdots \implies R$

$\neg P \implies Q_1 \cdots \implies \neg R$

$\neg P \implies R \land \neg R \equiv \text{False}$
Theorem: $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, $(\frac{a}{b})^2 \neq 2$.

A simple property (equality) should always “not” hold.

Proof by contradiction:

Theorem: $P$.

$\neg P \implies P_1 \cdots \implies R$

$\neg P \implies Q_1 \cdots \implies \neg R$

$\neg P \implies R \wedge \neg R \equiv \text{False}$

Contrapositive: $\text{True} \implies P$. 
Theorem: \( \sqrt{2} \) is irrational.

Must show: For every \( a, b \in \mathbb{Z} \), \( \left( \frac{a}{b} \right)^2 \neq 2 \).

A simple property (equality) should always “not” hold.

Proof by contradiction:

Theorem: \( P \).

\[ \neg P \implies P_1 \cdots \implies R \]
\[ \neg P \implies Q_1 \cdots \implies \neg R \]
\[ \neg P \implies R \land \neg R \equiv \text{False} \]

Contrapositive: \( \text{True} \implies P \). Theorem \( P \) is proven.
Proof by contradiction: form

**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, $(\frac{a}{b})^2 \neq 2$.

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem:** $P$.

$\neg P \implies P_1 \cdots \implies R$

$\neg P \implies Q_1 \cdots \implies \neg R$

$\neg P \implies R \land \neg R \equiv \text{False}$

Contrapositive: **True** $\implies P$. Theorem $P$ is proven.
Theorem: $\sqrt{2}$ is irrational.
Contradiction

**Theorem:** \( \sqrt{2} \) is irrational.

Assume \( \neg P \):

\[ \sqrt{2} = \frac{a}{b} \text{ for } a, b \in \mathbb{Z}. \]

Reduced form: \( a \) and \( b \) have no common factors.

\[ \sqrt{2}b = a^2b^2 = 4k^2 \]

\( a^2 \) is even \( \Rightarrow a \) is even.

\[ a = 2k \text{ for some integer } k. \]

\[ b^2 = 2k^2 \]

\( b^2 \) is even \( \Rightarrow b \) is even.

\( a \) and \( b \) have a common factor.

Contradiction.
Theorem: $\sqrt{2}$ is irrational.

Assume $\neg P$: $\sqrt{2} = a/b$ for $a, b \in \mathbb{Z}$. 

Contradiction.
Theorem: \( \sqrt{2} \) is irrational.

Assume \( \neg P: \sqrt{2} = a/b \) for \( a, b \in \mathbb{Z} \).

Reduced form: \( a \) and \( b \) have no common factors.
Theorem: $\sqrt{2}$ is irrational.

Assume \( \neg P: \sqrt{2} = \frac{a}{b} \) for \( a, b \in \mathbb{Z} \).

Reduced form: \( a \) and \( b \) have no common factors.

\[ \sqrt{2}b = a \]
Theorem: $\sqrt{2}$ is irrational.

Assume $\neg P$: $\sqrt{2} = a/b$ for $a, b \in \mathbb{Z}$.

Reduced form: $a$ and $b$ have no common factors.

$$\sqrt{2}b = a$$

$$2b^2 = a^2$$

Contradiction.
Contradiction

**Theorem:** $\sqrt{2}$ is irrational.

Assume $\neg P$: $\sqrt{2} = a/b$ for $a, b \in \mathbb{Z}$.

Reduced form: $a$ and $b$ have no common factors.

\[
\sqrt{2}b = a
\]

\[
2b^2 = a^2
\]

$a^2$ is even $\implies$ $a$ is even.
Contradiction

**Theorem:** \( \sqrt{2} \) is irrational.

Assume \( \neg P: \sqrt{2} = a/b \) for \( a, b \in \mathbb{Z} \).

Reduced form: \( a \) and \( b \) have no common factors.

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\( a^2 \) is even \( \implies \) \( a \) is even.

\( a = 2k \) for some integer \( k \)
Theorem: $\sqrt{2}$ is irrational.

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Theorem: \( \sqrt{2} \) is irrational.

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\( a = 2k \) for some integer \( k \)

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b^2 = 2k^2
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\( b^2 \) is even \( \implies \) \( b \) is even.
Theorem: \( \sqrt{2} \) is irrational.

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Reduced form: \( a \) and \( b \) have no common factors.

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\sqrt{2}b = a
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\( a^2 \) is even \( \implies \) \( a \) is even.

\( a = 2k \) for some integer \( k \)

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\( b^2 \) is even \( \implies \) \( b \) is even.

\( a \) and \( b \) have a common factor.
**Theorem:** $\sqrt{2}$ is irrational.

Assume $\neg P$: $\sqrt{2} = a/b$ for $a, b \in \mathbb{Z}$.

Reduced form: $a$ and $b$ have no common factors.

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\sqrt{2}b = a
\]

\[
2b^2 = a^2 = 4k^2
\]

$a^2$ is even $\implies$ $a$ is even.

$a = 2k$ for some integer $k$

\[
b^2 = 2k^2
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$b^2$ is even $\implies$ $b$ is even.

$a$ and $b$ have a common factor. Contradiction.
**Theorem:** \( \sqrt{2} \) is irrational.

Assume \( \neg P: \sqrt{2} = a/b \) for \( a, b \in \mathbb{Z} \).

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\( a^2 \) is even \( \implies \) \( a \) is even.

\( a = 2k \) for some integer \( k \)

\[
b^2 = 2k^2
\]

\( b^2 \) is even \( \implies \) \( b \) is even.

\( a \) and \( b \) have a common factor. Contradiction.
Proof by contradiction: example

**Theorem:** There are infinitely many primes.
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

Assume finitely many primes: \( p_1, \ldots, p_k \).

Consider \( q = (p_1 \times p_2 \times \cdots p_k) + 1 \).

\( q \) cannot be one of the primes as it is larger than any \( p_i \).

\( q \) has prime divisor \( p \) (\( p > 1 \) = R) which is one of \( p_i \).

\( p \) divides both \( x = p_1 \cdot p_2 \cdot \cdots p_k \) and \( q \),
and divides \( x - q \), \( \Rightarrow p \mid x - q \Rightarrow p \leq x - q = 1 \).

\( \Rightarrow p \leq 1. \) (Contradicts R.)

The original assumption that "the theorem is false" is false, thus the theorem is proven.
Proof by contradiction: example

Theorem: There are infinitely many primes.

Proof:

- Assume finitely many primes: \( p_1, \ldots, p_k \).
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

- Assume finitely many primes: $p_1, \ldots, p_k$.
- Consider $q = (p_1 \times p_2 \times \cdots p_k) + 1$.

$q$ cannot be one of the primes as it is larger than any $p_i$.
$q$ has prime divisor $p$ ("$p > 1$ = R") which is one of $p_i$.
$p$ divides both $x = p_1 \cdot p_2 \cdots p_k$ and $q$,
and divides $x - q$,
$⇒ p | x - q$,
$⇒ p \leq x - q = 1$.
so $p \leq 1$.
(Contradicts R).

The original assumption that "the theorem is false" is false,
thus the theorem is proven.
**Proof by contradiction: example**

**Theorem:** There are infinitely many primes.

**Proof:**

- Assume finitely many primes: $p_1, \ldots, p_k$.
- Consider
  \[
  q = (p_1 \times p_2 \times \cdots \times p_k) + 1.
  \]
- $q$ cannot be one of the primes as it is larger than any $p_i$. 

The original assumption that “the theorem is false” is false, thus the theorem is proven.
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

- Assume finitely many primes: \( p_1, \ldots, p_k \).
- Consider \( q = (p_1 \times p_2 \times \cdots p_k) + 1 \).
- \( q \) cannot be one of the primes as it is larger than any \( p_i \).
- \( q \) has prime divisor \( p \) ("\( p > 1 \) = R") which is one of \( p_i \).
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

- Assume finitely many primes: $p_1, \ldots, p_k$.
- Consider $q = (p_1 \times p_2 \times \cdots p_k) + 1$.

- $q$ cannot be one of the primes as it is larger than any $p_i$.
- $q$ has prime divisor $p$ ("$p > 1$" = R) which is one of $p_i$.
- $p$ divides both $x = p_1 \cdot p_2 \cdots p_k$ and $q$, thus the theorem is proven.
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

- Assume finitely many primes: \( p_1, \ldots, p_k \).
- Consider \( q = (p_1 \times p_2 \times \cdots \times p_k) + 1 \).
- \( q \) cannot be one of the primes as it is larger than any \( p_i \).
- \( q \) has prime divisor \( p \) ("\( p > 1 \) = R \) which is one of \( p_i \).
- \( p \) divides both \( x = p_1 \cdot p_2 \cdot \cdots \cdot p_k \) and \( q \), and divides \( x - q \),

The original assumption that "the theorem is false" is false, thus the theorem is proven.
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

- Assume finitely many primes: $p_1, \ldots, p_k$.
- Consider

$$q = (p_1 \times p_2 \times \cdots p_k) + 1.$$ 

- $q$ cannot be one of the primes as it is larger than any $p_i$.
- $q$ has prime divisor $p$ ("$p > 1$" = R) which is one of $p_i$.
- $p$ divides both $x = p_1 \cdot p_2 \cdots p_k$ and $q$, and divides $x - q$.
- $\implies p | x - q$
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

- Assume finitely many primes: $p_1, \ldots, p_k$.
- Consider
  
  $$q = (p_1 \times p_2 \times \cdots p_k) + 1.$$

- $q$ cannot be one of the primes as it is larger than any $p_i$.
- $q$ has prime divisor $p$ ("$p > 1$" = R) which is one of $p_i$.
- $p$ divides both $x = p_1 \cdot p_2 \cdots p_k$ and $q$, and divides $x - q$,
  
  $$\implies p|x - q \implies p \leq x - q$$

The original assumption that "the theorem is false" is false, thus the theorem is proven.
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

- Assume finitely many primes: \( p_1, \ldots, p_k \).
- Consider \( q = (p_1 \times p_2 \times \cdots \times p_k) + 1. \)
- \( q \) cannot be one of the primes as it is larger than any \( p_i \).
- \( q \) has prime divisor \( p \) ("\( p > 1 \)" = R) which is one of \( p_i \).
- \( p \) divides both \( x = p_1 \cdot p_2 \cdots p_k \) and \( q \), and divides \( x - q \),
- \( \implies p | x - q \implies p \leq x - q = 1. \)
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

- Assume finitely many primes: \( p_1, \ldots, p_k \).
- Consider
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  q = (p_1 \times p_2 \times \cdots p_k) + 1.
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- \( q \) cannot be one of the primes as it is larger than any \( p_i \).
- \( q \) has prime divisor \( p \) ("\( p > 1 \) = R") which is one of \( p_i \).
- \( p \) divides both \( x = p_1 \cdot p_2 \cdot \cdots p_k \) and \( q \), and divides \( x - q \),
  \[
  \implies p | x - q \implies p \leq x - q = 1.
  \]
- so \( p \leq 1 \).

The original assumption that "the theorem is false" is false, thus the theorem is proven.
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

- Assume finitely many primes: $p_1, \ldots, p_k$.
- Consider
  
  $$q = (p_1 \times p_2 \times \cdots p_k) + 1.$$  

- $q$ cannot be one of the primes as it is larger than any $p_i$.
- $q$ has prime divisor $p$ ("$p > 1$" = R ) which is one of $p_i$.
- $p$ divides both $x = p_1 \cdot p_2 \cdots p_k$ and $q$, and divides $x - q$,
  
  $$\implies p|x - q \implies p \leq x - q = 1.$$  

- so $p \leq 1$. (**Contradicts R.**)
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

- **Assume finitely many primes:** $p_1, \ldots, p_k$.
- **Consider**
  \[ q = (p_1 \times p_2 \times \cdots p_k) + 1. \]

- $q$ cannot be one of the primes as it is larger than any $p_i$.
- $q$ has prime divisor $p$ (”$p > 1$” = R) which is one of $p_i$.
- $p$ divides both $x = p_1 \cdot p_2 \cdots p_k$ and $q$, and divides $x - q$,
  \[ \implies p \mid x - q \implies p \leq x - q = 1. \]
- so $p \leq 1$. (Contradicts $R$.)

The original assumption that “the theorem is false” is false, thus the theorem is proven.
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

- Assume finitely many primes: \( p_1, \ldots, p_k \).
- Consider \( q = (p_1 \times p_2 \times \cdots p_k) + 1 \).

- \( q \) cannot be one of the primes as it is larger than any \( p_i \).
- \( q \) has prime divisor \( p \) ("\( p > 1 \)" = \( R \)) which is one of \( p_i \).
- \( p \) divides both \( x = p_1 \cdot p_2 \cdots p_k \) and \( q \), and divides \( x - q \),
- \( \implies p \mid x - q \implies p \leq x - q = 1. \)
- so \( p \leq 1 \). (Contradicts \( R \).)

The original assumption that “the theorem is false” is false, thus the theorem is proven.
Did we prove?

- “The product of the first $k$ primes plus 1 is prime.”
Did we prove?

▶ “The product of the first $k$ primes plus 1 is prime.”
▶ No.
Did we prove?

▶ “The product of the first $k$ primes plus 1 is prime.”
▶ No.
▶ The chain of reasoning started with a false statement.
Did we prove?

- “The product of the first $k$ primes plus 1 is prime.”
- No.
- The chain of reasoning started with a false statement.

Consider example..
Product of first $k$ primes..

Did we prove?

- “The product of the first $k$ primes plus 1 is prime.”
- No.
- The chain of reasoning started with a false statement.

Consider example..

- $2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30031 = 59 \times 509$
Product of first $k$ primes..

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- $2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30031 = 59 \times 509$
- There is a prime in between 13 and $q = 30031$ that divides $q$. 
Did we prove?

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Consider example..

- $2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30031 = 59 \times 509$
- There is a prime in between 13 and $q = 30031$ that divides $q$.
- Proof assumed no primes in between $p_k$ and $q$. 
Proof by cases.

**Theorem:** \( x^5 - x + 1 = 0 \) has no solution in the rationals.
Proof by cases.

**Theorem:** $x^5 - x + 1 = 0$ has no solution in the rationals.

**Proof:** First a lemma...

---

**Lemma:** If $x$ is a solution to $x^5 - x + 1 = 0$ and $x = a/b$ for $a, b \in \mathbb{Z}$, then both $a$ and $b$ are even.

**Reduced form $a/b$:** $a$ and $b$ can't both be even!

**Proof of lemma:**

Assume a solution of the form $a/b$.

$$(a/b)^5 - a/b + 1 = 0$$

Multiply by $b^5$,

$$a^5 - ab^4 + b^5 = 0$$

**Case 1:** $a$ odd, $b$ odd: odd - odd + odd = even. Not possible.

**Case 2:** $a$ even, $b$ odd: even - even + odd = even. Not possible.

**Case 3:** $a$ odd, $b$ even: odd - even + even = even. Not possible.

**Case 4:** $a$ even, $b$ even: even - even + even = even. Possible.

The fourth case is the only one possible, so the lemma follows.
Proof by cases.

**Theorem:** \(x^5 - x + 1 = 0\) has no solution in the rationals.

**Proof:** First a lemma...

**Lemma:** If \(x\) is a solution to \(x^5 - x + 1 = 0\) and \(x = \frac{a}{b}\) for \(a, b \in \mathbb{Z}\), then both \(a\) and \(b\) are even.
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\begin{align*}
    a^5 - ab^4 + b^5 &= 0 \\
    \text{Case 1: } a \text{ odd, } b \text{ odd: odd - odd + odd = even. Not possible.} \\
    \text{Case 2: } a \text{ even, } b \text{ odd: even - even + odd = even. Not possible.} \\
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**Proof of lemma:** Assume a solution of the form \( \frac{a}{b} \).

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\left( \frac{a}{b} \right)^5 - \frac{a}{b} + 1 = 0
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Proof by cases.

**Theorem:** There exist irrational \( x \) and \( y \) such that \( x^y \) is rational.

Let \( x = y = \sqrt{2} \).

Case 1: \( x^y = \sqrt{2}^\sqrt{2} \) is rational. 

\[ x^y = (\sqrt{2})^{\sqrt{2}} \]

Case 2: \( \sqrt{2}^\sqrt{2} \) is irrational.

▶ New values: \( x = \sqrt{2}^\sqrt{2}, \ y = \sqrt{2} \).

\[ x^y = (\sqrt{2}^\sqrt{2})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}} = \sqrt{2}^2 = 2 \]

Thus, we have irrational \( x \) and \( y \) with a rational \( x^y \) (i.e., 2). 

One of the cases is true so theorem holds.

Question: Which case holds?

Don't know!!
Proof by cases.

**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational.

Let $x = y = \sqrt{2}$.

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Thus, we have irrational $x$ and $y$ with a rational $x^y$ (i.e., $\sqrt{2}$).

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$$x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}}$$

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One of the cases is true so theorem holds.

Question: Which case holds? Don’t know!!!
Be careful.

**Theorem:** $3 = 4$

Proof: Assume $3 = 4$. Start with $12 = 12$. Divide one side by 3 and the other by 4 to get $4 = 3$. By commutativity the theorem holds. Don't assume what you want to prove!
Be careful.

**Theorem:** $3 = 4$

**Proof:** Assume $3 = 4$. 

Don't assume what you want to prove!
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**Theorem:** $3 = 4$

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Start with $12 = 12$. 
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By commutativity
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**Proof:** Assume 3 = 4.

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Start with $12 = 12$.

Divide one side by 3 and the other by 4 to get $4 = 3$.

By commutativity theorem holds. □
Theorem: $3 = 4$

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By commutativity theorem holds.

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**Theorem:** $1 = 2$

**Proof:**
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**Theorem:** $1 = 2$

**Proof:** For $x = y$, we have
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**Theorem:** $1 = 2$

**Proof:** For $x = y$, we have

$$(x^2 - xy) = x^2 - y^2$$
Be really careful!

**Theorem**: $1 = 2$

**Proof**: For $x = y$, we have

$$(x^2 - xy) = x^2 - y^2$$

$$x(x - y) = (x + y)(x - y)$$

Dividing by zero is no good.

Also: Multiplying inequalities by a negative. $P \nRightarrow Q$ does not mean $Q \nRightarrow P$. 
Be really careful!

**Theorem: 1 = 2**

**Proof:** For $x = y$, we have

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x(x - y) = (x + y)(x - y)
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x = (x + y)
\]
Be really careful!

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**Theorem:** \(1 = 2\)

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1 = 2
\]
Dividing by zero is no good.

Also: Multiplying inequalities by a negative.

$P \implies Q$ does not mean $Q \implies P$. 
Summary: Note 2.

Direct Proof:
Summary: Note 2.

Direct Proof:
To Prove: \( P \implies Q \).
Summary: Note 2.

Direct Proof:
To Prove: $P \implies Q$. Assume $P$. 

Summary: Note 2.

Direct Proof:
To Prove: $P \implies Q$. Assume $P$. Prove $Q$. 

By Contraposition:

To Prove: $P \implies Q$. Assume $\neg Q$. Prove $\neg P$. 

By Contradiction:


By Cases: informal.

Universal: show that statement holds in all cases.
Existence: used cases where one is true.

Either $\sqrt{2}$ and $\sqrt{2}$ worked.

or $\sqrt{2}$ and $\sqrt{2}$ worked.

Careful when proving!

Don’t assume the theorem.
Divide by zero.
Watch converse.

... 

And finally.
Have a nice weekend!!
Summary: Note 2.

Direct Proof:
To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
Summary: Note 2.

Direct Proof:
  To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
  To Prove: $P \implies Q$

By Cases: informal.
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Either $\sqrt{2}$ and $\sqrt{2}$ worked.
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Careful when proving!
  Don’t assume the theorem.
  Divide by zero.
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And finally.

Have a nice weekend!!
Summary: Note 2.

Direct Proof:
To Prove: \( P \implies Q \). Assume \( P \). Prove \( Q \).

By Contraposition:
To Prove: \( P \implies Q \) Assume \( \neg Q \).

By Contradiction:
To Prove: \( P \) Assume \( \neg P \).

By Cases:
- Universal: show that statement holds in all cases.
- Existence: used cases where one is true.
  - \( \sqrt{2} \) and \( \sqrt{2} \) worked.
  - or \( \sqrt{2} \) and \( \sqrt{2} \) worked.

Careful when proving!
- Don't assume the theorem.
- Divide by zero.
- Watch converse.

And finally.
Have a nice weekend!!
Summary: Note 2.

Direct Proof:
To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
To Prove: $P \implies Q$ Assume $\neg Q$. Prove $\neg P$.
Summary: Note 2.

Direct Proof:
   To Prove: $P \Rightarrow Q$. Assume $P$. Prove $Q$.

By Contraposition:
   To Prove: $P \Rightarrow Q$ Assume $\neg Q$. Prove $\neg P$.

By Contradiction:
Summary: Note 2.

Direct Proof:
To Prove: \( P \implies Q \). Assume \( P \). Prove \( Q \).

By Contraposition:
To Prove: \( P \implies Q \). Assume \( \neg Q \). Prove \( \neg P \).

By Contradiction:
To Prove: \( P \)
Summary: Note 2.

Direct Proof:
To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
To Prove: $P \implies Q$ Assume $\neg Q$. Prove $\neg P$.

By Contradiction:
To Prove: $P$ Assume $\neg P$.

...
Summary: Note 2.

Direct Proof:
To Prove: \( P \implies Q \). Assume \( P \). Prove \( Q \).

By Contraposition:
To Prove: \( P \implies Q \) Assume \( \neg Q \). Prove \( \neg P \).

By Contradiction:
To Prove: \( P \) Assume \( \neg P \). Prove \text{False}.
Direct Proof:
   To Prove: \( P \implies Q \). Assume \( P \). Prove \( Q \).

By Contraposition:
   To Prove: \( P \implies Q \) Assume \( \neg Q \). Prove \( \neg P \).

By Contradiction:
   To Prove: \( P \) Assume \( \neg P \). Prove \( \text{False} \).

By Cases: informal.
Summary: Note 2.

Direct Proof:
To Prove: \( P \implies Q \). Assume \( P \). Prove \( Q \).

By Contraposition:
To Prove: \( P \implies Q \) Assume \( \neg Q \). Prove \( \neg P \).

By Contradiction:
To Prove: \( P \) Assume \( \neg P \). Prove False.

By Cases: informal.
Universal: show that statement holds in all cases.
Direct Proof:
   To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
   To Prove: $P \implies Q$ Assume $\neg Q$. Prove $\neg P$.

By Contradiction:
   To Prove: $P$ Assume $\neg P$. Prove False.

By Cases: informal.
   Universal: show that statement holds in all cases.
   Existence: used cases where one is true.

Careful when proving!
Don't assume the theorem.
Divide by zero.
Watch converse.

And finally.
Have a nice weekend!!
Summary: Note 2.

Direct Proof:
To Prove: \( P \implies Q \). Assume \( P \). Prove \( Q \).

By Contraposition:
To Prove: \( P \implies Q \) Assume \( \neg Q \). Prove \( \neg P \).

By Contradiction:
To Prove: \( P \) Assume \( \neg P \). Prove False .

By Cases: informal.
Universal: show that statement holds in all cases.
Existence: used cases where one is true.
Either \( \sqrt{2} \) and \( \sqrt{2} \) worked.

Careful when proving!
Don't assume the theorem.
Divide by zero.
Watch converse.

...And finally.
Have a nice weekend!!
Summary: Note 2.

Direct Proof:
To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
To Prove: $P \implies Q$ Assume $\neg Q$. Prove $\neg P$.

By Contradiction:
To Prove: $P$ Assume $\neg P$. Prove False.

By Cases: informal.
Universal: show that statement holds in all cases.
Existence: used cases where one is true.
   Either $\sqrt{2}$ and $\sqrt{2}$ worked.
   or $\sqrt{2}$ and $\sqrt{2} \sqrt{2}$ worked.

Careful when proving! Don’t assume the theorem. Divide by zero. Watch converse.

And finally. Have a nice weekend!!
Summary: Note 2.

Direct Proof:
To Prove: \( P \implies Q \). Assume \( P \). Prove \( Q \).

By Contraposition:
To Prove: \( P \implies Q \) Assume \( \neg Q \). Prove \( \neg P \).

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Universal: show that statement holds in all cases.
Existence: used cases where one is true.
   Either \( \sqrt{2} \) and \( \sqrt{2} \) worked.
   or \( \sqrt{2} \) and \( \sqrt{2\sqrt{2}} \) worked.
Summary: Note 2.

Direct Proof:
To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
To Prove: $P \implies Q$ Assume $\neg Q$. Prove $\neg P$.

By Contradiction:
To Prove: $P$ Assume $\neg P$. Prove $\text{False}$.

By Cases: informal.
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   Either $\sqrt{2}$ and $\sqrt{2}$ worked.
   or $\sqrt{2}$ and $\sqrt{2}^{\sqrt{2}}$ worked.

Careful when proving!
Summary: Note 2.

Direct Proof:
To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
To Prove: $P \implies Q$ Assume $\neg Q$. Prove $\neg P$.

By Contradiction:
To Prove: $P$ Assume $\neg P$. Prove False.

By Cases: informal.
Universal: show that statement holds in all cases.
Existence: used cases where one is true.
Either $\sqrt{2}$ and $\sqrt{2}$ worked.
  or $\sqrt{2}$ and $\sqrt{2^\sqrt{2}}$ worked.

Careful when proving!
Don’t assume the theorem.
Summary: Note 2.

Direct Proof:
To Prove: \( P \implies Q \). Assume \( P \). Prove \( Q \).

By Contraposition:
To Prove: \( P \implies Q \). Assume \( \neg Q \). Prove \( \neg P \).

By Contradiction:
To Prove: \( P \). Assume \( \neg P \). Prove False.

By Cases: informal.
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Existence: used cases where one is true.
   Either \( \sqrt{2} \) and \( \sqrt{2} \) worked.
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Careful when proving!
Don’t assume the theorem. Divide by zero.
Summary: Note 2.

Direct Proof:
To Prove: \( P \implies Q \). Assume \( P \). Prove \( Q \).

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To Prove: \( P \implies Q \) Assume \( \neg Q \). Prove \( \neg P \).

By Contradiction:
To Prove: \( P \) Assume \( \neg P \). Prove false.

By Cases: informal.
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   Either \( \sqrt{2} \) and \( \sqrt{2} \) worked.
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Careful when proving!
Don’t assume the theorem. Divide by zero. Watch converse.
Summary: Note 2.

Direct Proof:
To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
To Prove: $P \implies Q$ Assume $\neg Q$. Prove $\neg P$.

By Contradiction:
To Prove: $P$ Assume $\neg P$. Prove False.

By Cases: informal.
Universal: show that statement holds in all cases.
Existence: used cases where one is true.
   Either $\sqrt{2}$ and $\sqrt{2}$ worked.
     or $\sqrt{2}$ and $\sqrt{2}^{\sqrt{2}}$ worked.

Careful when proving!
Don’t assume the theorem. Divide by zero. Watch converse. ...
Summary: Note 2.

Direct Proof:
To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
To Prove: $P \implies Q$ Assume $\neg Q$. Prove $\neg P$.

By Contradiction:
To Prove: $P$ Assume $\neg P$. Prove False.

By Cases: informal.
Universal: show that statement holds in all cases.
Existence: used cases where one is true.
  Either $\sqrt{2}$ and $\sqrt{2}$ worked.
  or $\sqrt{2}$ and $\sqrt{2}^2$ worked.

Careful when proving!
  Don’t assume the theorem. Divide by zero. Watch converse. ...

And finally.
Summary: Note 2.

Direct Proof:
To Prove: \( P \implies Q \). Assume \( P \). Prove \( Q \).

By Contraposition:
To Prove: \( P \implies Q \) Assume \( \neg Q \). Prove \( \neg P \).

By Contradiction:
To Prove: \( P \) Assume \( \neg P \). Prove \textbf{False}.

By Cases: informal.

Universal: show that statement holds in all cases.
Existence: used cases where one is true.

Either \( \sqrt{2} \) and \( \sqrt{2} \) worked.

or \( \sqrt{2} \) and \( \sqrt{2}^{\sqrt{2}} \) worked.

Careful when proving!

\textbf{Don’t assume the theorem. Divide by zero. Watch converse.} ...

And finally. \textbf{Have a nice weekend!!}
CS70: Note 3. Induction!

1. The natural numbers.
2. 5 year old Gauss.
3. ..and Induction.
4. Simple Proof.
The naturals.
The naturals.
The naturals.
The naturals.

0, 1,
The naturals.

0, 1, 2,
The naturals.

0, 1, 2, 3,
The naturals.

0, 1, 2, 3,
...,.
The naturals.

$0, 1, 2, 3, \ldots, n,$
The naturals.

0, 1, 2, 3, ...

n, n + 1,
The naturals.

0, 1, 2, 3,
..., n, n+1, n+2, n+3,
The naturals.

0, 1, 2, 3,
..., n, n+1, n+2, n+3, ...

\[
\begin{array}{c}
0 \\
1 \\
2 \\
3 \\
n \ \\
n+1 \\
n+2 \\
n+3 \\
\vdots \\
\end{array}
\]
A formula.

Teacher: Hello class.

Teacher: Please add the numbers from 1 to 100.

Gauss: It's \((100)(101)\) or 5050!
Teacher: Hello class.
Teacher: Hello class.
Teacher:
A formula.

Teacher: Hello class.
Teacher: *Please add the numbers from 1 to 100.*
Teacher: Hello class.
Teacher: *Please add the numbers from 1 to 100.*
Gauss: It’s
Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.
Gauss: It's \( \frac{(100)(101)}{2} \)
Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.
Gauss: It’s $\frac{(100)(101)}{2}$ or 5050!
Child Gauss: $(\forall n \in \mathbb{N})\left(\sum_{i=1}^{n} i = \frac{n(n+1)}{2}\right)$
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Proof?

Idea: assume predicate \(P(n)\) for \(n = k\).

\(P(k)\) is \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate, \(P(n)\) true for \(n = k+1\)?

\(\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}\).

How about \(k+2\).

Same argument starting at \(k+1\) works!

Induction Step.

\(P(k) = \Rightarrow P(k+1)\).

Is this a proof?

It shows that we can always move to the next step.

Need to start somewhere.

\(P(0)\) is \(\sum_{i=1}^{0} i = \frac{0(0+1)}{2}\) Base Case.

Statement is true for \(n = 0\)

\(P(0)\) is true plus inductive step \(\Rightarrow \) true for \(n = 1\)

\(P(0) \wedge (P(0) = \Rightarrow P(1)) = \Rightarrow P(1)\) plus inductive step \(\Rightarrow \) true for \(n = 2\)

... true for \(n = k\) \(\Rightarrow \) true for \(n = k+1\)

Predicate, \(P(n)\), True for all natural numbers!

Proof by Induction.
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate \(P(n)\) for \(n = k\).
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate \(P(n)\) for \(n = k\). \(P(k)\) is \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate \(P(n)\) for \(n = k\). \(P(k)\) is \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate, \(P(n)\) true for \(n = k + 1\)?
Child Gauss: $(\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k + 1$?

$\sum_{i=1}^{k+1} i$
Gauss and Induction

Child Gauss: \((\forall \, n \in \mathbb{N}) \left( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \right)\) Proof?

Idea: assume predicate \(P(n)\) for \(n = k\). \(P(k)\) is \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate, \(P(n)\) true for \(n = k + 1\)?

\[ \sum_{i=1}^{k+1} i = \left( \sum_{i=1}^{k} i \right) + (k + 1) \]
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate \(P(n)\) for \(n = k\). \(P(k)\) is \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate, \(P(n)\) true for \(n = k + 1\)?

\[\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1\]
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N}) (\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

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Is predicate, \(P(n)\) true for \(n = k + 1\)?

\[
\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.
\]
Gauss and Induction

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\]

How about \(k + 2\).
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N}) (\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate \(P(n)\) for \(n = k\). \(P(k)\) is \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

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How about \(k + 2\). Same argument starting at \(k + 1\) works!
Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate \(P(n)\) for \(n = k\). \(P(k)\) is \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

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\]

How about \(k + 2\). Same argument starting at \(k + 1\) works!

**Induction Step.**
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate \(P(n)\) for \(n = k\). \(P(k)\) is \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate, \(P(n)\) true for \(n = k + 1\)?

\[
\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.
\]

How about \(k + 2\). Same argument starting at \(k + 1\) works! 

**Induction Step.** \(P(k) \implies P(k+1)\).
Gauss and Induction

Child Gauss: \( (\forall n \in \mathbb{N}) (\sum_{i=1}^{n} i = \frac{n(n+1)}{2} ) \) Proof?

Idea: assume predicate \( P(n) \) for \( n = k \). \( P(k) \) is \( \sum_{i=1}^{k} i = \frac{k(k+1)}{2} \).

Is predicate, \( P(n) \) true for \( n = k + 1 \)?

\[
\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.
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**Induction Step.** \( P(k) \implies P(k+1) \).

Is this a proof?
Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

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Is predicate, \(P(n)\) true for \(n = k + 1\)?

\[
\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.
\]

How about \(k + 2\). Same argument starting at \(k + 1\) works!

**Induction Step.** \(P(k) \implies P(k + 1).\)

Is this a proof? It shows that we can always move to the next step.
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate \(P(n)\) for \(n = k\). \(P(k)\) is \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

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\]

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**Induction Step.** \(P(k) \implies P(k + 1).\)

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere.
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate \(P(n)\) for \(n = k\). \(P(k)\) is \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

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\[\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.\]

How about \(k + 2\). Same argument starting at \(k + 1\) works!

**Induction Step.** \(P(k) \implies P(k+1)\).

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \(P(0)\) is \(\sum_{i=0}^{0} i = 1 = \frac{(0)(0+1)}{2}\)
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate \(P(n)\) for \(n = k\). \(P(k)\) is \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

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Child Gauss: $\forall n \in \mathbb{N}(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

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plus inductive step
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\[
\ldots
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\[
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Predicate, $P(n)$, True for all natural numbers!
**Gauss and Induction**

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\[\ldots\]

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\[\ldots\]

Predicate, \(P(n)\), **True** for all natural numbers! **Proof by Induction.**