CS70: Lecture 20.

Distributions; Independent RVs

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1. Review: Expectation
2. Distributions
3. Independent RVs

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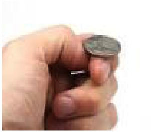

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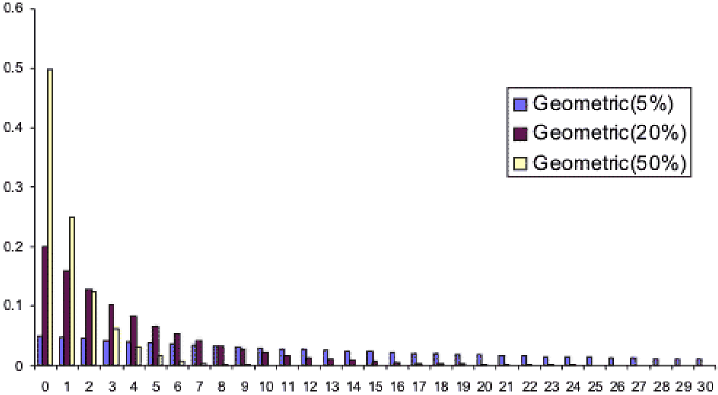

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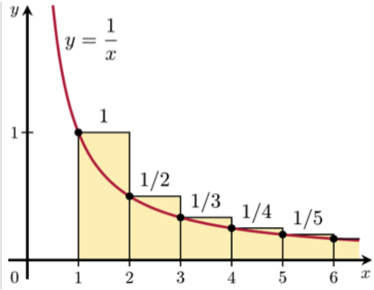

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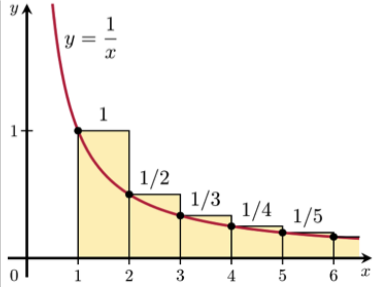

A good approximation is

$$H(n) \approx \ln(n) + \gamma \text{ where } \gamma \approx 0.58 \text{ (Euler-Mascheroni constant).}$$

Harmonic sum: Paradox

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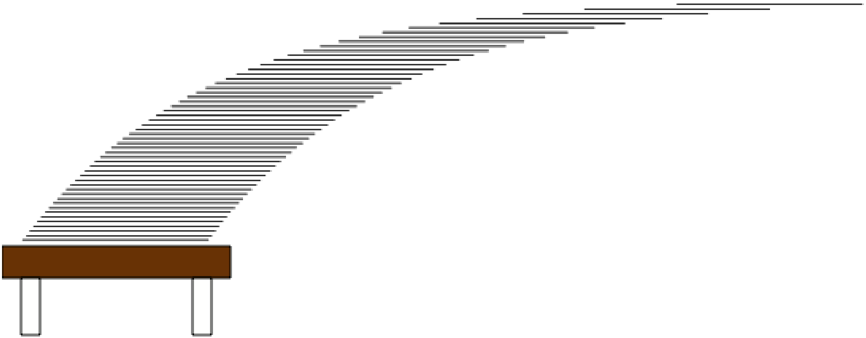

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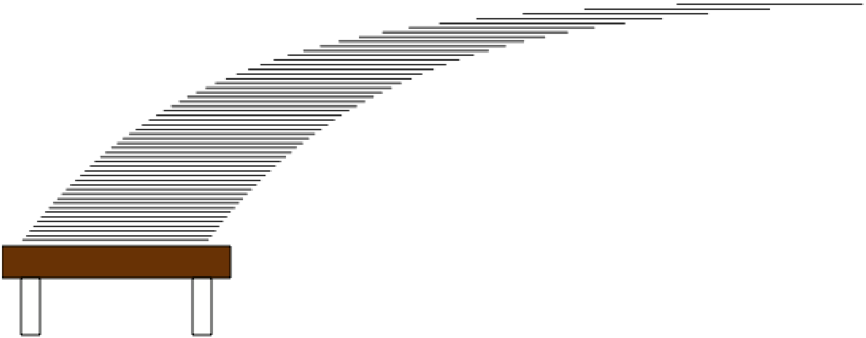

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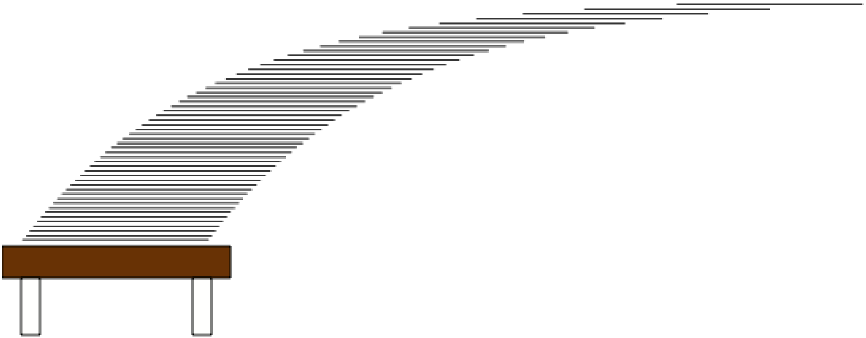


If each card has length 2, the stack can extend $H(n)$ to the right of the table. As n increases, you can go as far as you want!

Paradox

par·a·dox

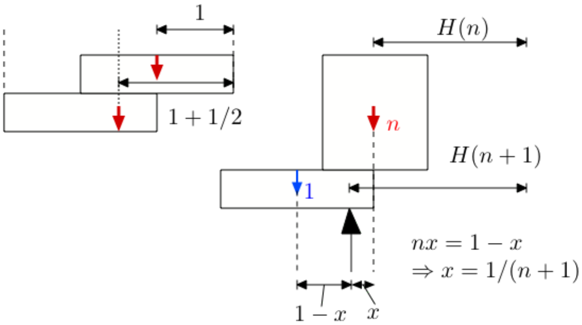
/ˈperəˌdäks/

noun

a statement or proposition that, despite sound (or apparently sound) reasoning from acceptable premises, leads to a conclusion that seems senseless, logically unacceptable, or self-contradictory.
"a potentially serious conflict between quantum mechanics and the general theory of relativity known as the information paradox"

- a seemingly absurd or self-contradictory statement or proposition that when investigated or explained may prove to be well founded or true.
"in a paradox, he has discovered that stepping back from his job has increased the rewards he gleans from it"
synonyms: contradiction, contradiction in terms, self-contradiction, inconsistency, incongruity; More
- a situation, person, or thing that combines contradictory features or qualities.
"the mingling of deciduous trees with elements of desert flora forms a fascinating ecological paradox"

Stacking



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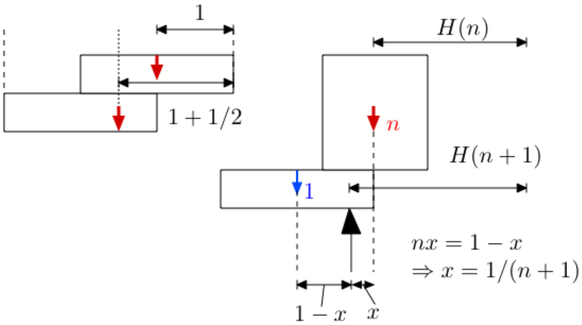

The cards have width 2.

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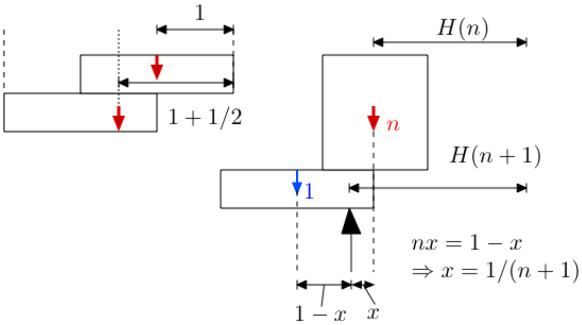

The cards have width 2. Induction shows that the center of gravity after n cards is $H(n)$ away from the right-most edge.

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Let X be $G(p)$. Then, for $n \geq 0$,

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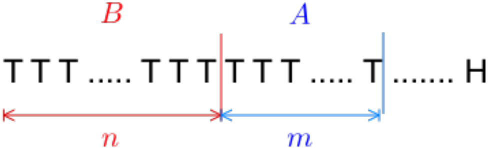

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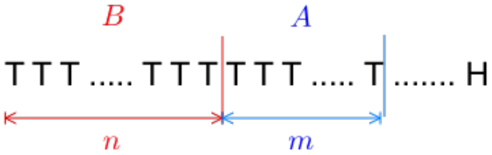

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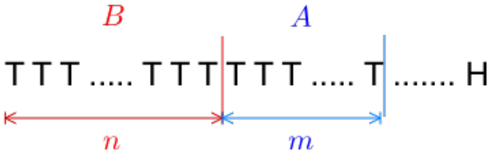

$$Pr[X > n+m | X > n] = Pr[A|B] = Pr[A] = Pr[X > m].$$

The coin is memoryless, therefore, so is X.

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Theorem: For a r.v. X that takes the values $\{0,1,2,\ldots\}$, one has

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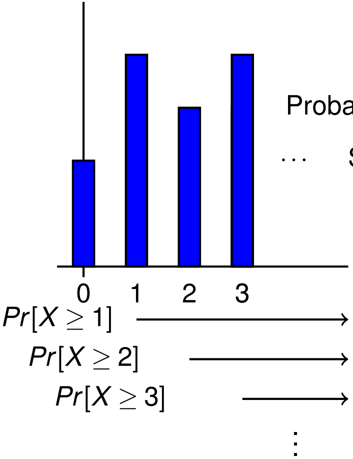

Probability mass at i, counted i times.

Same as $\sum_{i=1}^{\infty} i \times Pr[X = i]$.

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Experiment: flip a coin n times. The coin is such that $Pr[H] = \lambda/n$.
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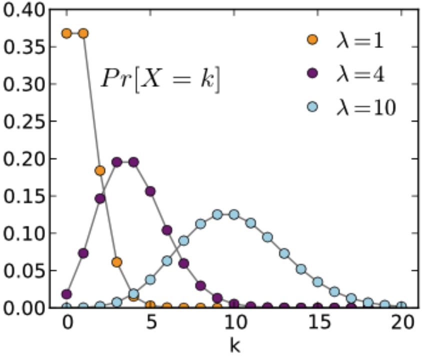

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□

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Equal Time: B. Geometric

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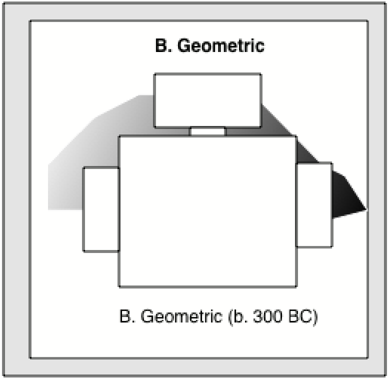

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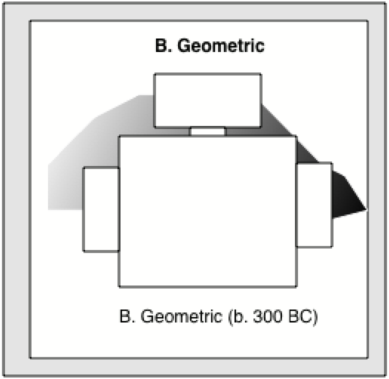

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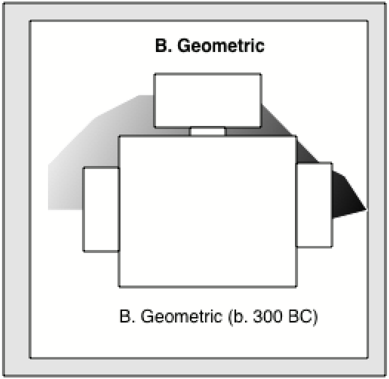

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& \quad = Pr[(X_1, \ldots, X_4) \in B_1, (X_5, \ldots, X_8) \in B_2, (X_9, \ldots, X_{11}) \in B_3] \\
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\end{aligned}$$

Functions of mutually independent RVs

One has the following result:

Theorem

Functions of disjoint collections of mutually independent random variables are mutually independent.

Example:

Let $\{X_n, n \geq 1\}$ be mutually independent. Then,

$Y_1 := X_1 X_2 (X_3 + X_4)^2, Y_2 := \max\{X_5, X_6\} - \min\{X_7, X_8\}, Y_3 := X_9 \cos(X_{10} + X_{11})$ are mutually independent.

Proof:

Let $B_1 := \{(x_1, x_2, x_3, x_4) \mid x_1 x_2 (x_3 + x_4)^2 \in A_1\}$. Similarly for B_2, B_2. Then

$$
\begin{aligned}
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$1_{A \Delta B} = f(1_A, 1_B)$ where
$f(0,0) = 0, f(1,0) = 1, f(0,1) = 1, f(1,1) = 0$

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Hence, $1_{A \Delta B}, 1_{C \setminus D}, 1_{\bar{E}}$ are functions of mutually independent RVs.

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