Distributions; Independent RVs
Distributions; Independent RVs
1. Review: Expectation
2. Distributions
3. Independent RVs
Review: Expectation
Review: Expectation

\[ E[X] := \sum_x x \Pr[X = x] = \sum_\omega X(\omega) \Pr[\omega]. \]
Review: Expectation

- $E[X] := \sum_x x Pr[X = x] = \sum_\omega X(\omega) Pr[\omega]$.

- $E[g(X, Y)] = \sum_{x,y} g(x, y) Pr[X = x, Y = y]$
Review: Expectation

- \( E[X] := \sum_x xPr[X = x] = \sum_\omega X(\omega)Pr[\omega]. \)

- \( E[g(X, Y)] = \sum_{x,y} g(x, y)Pr[X = x, Y = y] \)
  \[ = \sum_\omega g(X(\omega), Y(\omega))Pr[\omega] \]
Review: Expectation

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\( E[g(X, Y)] = \sum_{x,y} g(x, y)Pr[X = x, Y = y] \)
\( = \sum_\omega g(X(\omega), Y(\omega))Pr[\omega] \)

\( E[aX + bY + c] = aE[X] + bE[Y] + c. \)
Uniform Distribution

Roll a six-sided balanced die. Let $X$ be the number of pips (dots). Then $X$ is equally likely to take any of the values $\{1, 2, \ldots, 6\}$. We say that $X$ is uniformly distributed in $\{1, 2, \ldots, 6\}$.

More generally, we say that $X$ is uniformly distributed in $\{1, 2, \ldots, n\}$ if

$$\Pr[X = m] = \frac{1}{n}$$

for $m = 1, 2, \ldots, n$. In that case,

$$E[X] = \frac{n}{2} \sum_{m=1}^{n} \frac{m}{n} = \frac{1}{n} \sum_{m=1}^{n} m = \frac{n+1}{2}.$$
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Uniform Distribution

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More generally, we say that $X$ is uniformly distributed in $\{1, 2, \ldots, n\}$ if $Pr[X = m] = 1/n$ for $m = 1, 2, \ldots, n$. 

\[ E[X] = \sum_{m=1}^{n} m \cdot \frac{1}{n} = \frac{n}{2} + \frac{1}{2} \]
Uniform Distribution

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$$E[X] = \sum_{m=1}^{n} m Pr[X = m]$$
Uniform Distribution

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Geometric Distribution

Let's flip a coin with $\Pr[\text{H}] = p$ until we get $\text{H}$. For instance:

$\omega_1 = \text{H}$, or $\omega_2 = \text{T H}$, or $\omega_3 = \text{T T H}$, or $\omega_n = \text{T T T T \cdots T H}$.

Note that $\Omega = \{\omega_n, n = 1, 2, \ldots\}$.

Let $X$ be the number of flips until the first $\text{H}$. Then, $X(\omega_n) = n$.

Also, $\Pr[X = n] = (1 - p)^{n-1}p$, $n \geq 1$.
Geometric Distribution

Let’s flip a coin with $Pr[H] = p$ until we get $H$. 
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For instance:

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For instance:

\[ \omega_1 = H, \text{ or} \]
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$$Pr[X = n] = (1 - p)^{n-1} p, n \geq 1.$$  

Note that

$$\sum_{n=1}^{\infty} Pr[X_n] =$$
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S = 1 + a + a^2 + a^3 + \cdots
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\begin{array}{c}
aS \\
\end{array} & = a + a^2 + a^3 + a^4 + \cdots
\end{align*}
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**Geometric Distribution**

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Geometric Distribution: Expectation

\[ X =_{D} G(p), \text{ i.e., } Pr[X = n] = (1 - p)^{n-1}p, n \geq 1. \]
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\[ E[X] = \sum_{n=1}^{\infty} n Pr[X = n] = \sum_{n=1}^{\infty} n(1 - p)^{n-1}p. \]
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Thus,

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E[X] = p + 2(1 - p)p + 3(1 - p)^2 p + 4(1 - p)^3 p + \cdots
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by subtracting the previous two identities
Geometric Distribution: Expectation

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Hence,

\[ E[X] = \frac{1}{p}. \]
**Experiment**: Get coupons at random from \( n \) until collect all \( n \) coupons.
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**Outcomes:** \{123145..., 56765...\}
Coupon Collectors Problem.

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Time to collect coupons

\(X\)-time to get \(n\) coupons.
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$X_2$ - time to get second coupon after getting first.
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$Pr[\text{“get second coupon”|“got milk”}]$
Time to collect coupons

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$Pr["get second coupon"|"got first coupon"] = \frac{n-1}{n}$
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$E[X_2]$?
Time to collect coupons

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$E[X_2]$? Geometric
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\( E[X_2]? \) Geometric!
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$X$-time to get $n$ coupons.

$X_1$ - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

$X_2$ - time to get second coupon after getting first.

$Pr[\text{“get second coupon”|“got milk first coupon”}] = \frac{n-1}{n}$

$E[X_2]$? Geometric !!!
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$E[X_2]$? Geometric ! ! ! $\implies E[X_2] = \frac{1}{p} = \frac{1}{n-1} = \frac{n}{n-1}.$

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Pr[“get second coupon”|“got first coupon”] = \( \frac{n-1}{n} \)
E[X2]? Geometric ! ! ! \( \implies E[X_2] = \frac{1}{p} = \frac{1}{n-1} = \frac{n}{n-1}. \)
Pr[“getting ith coupon|“got i – 1rst coupons”] = \( \frac{n-(i-1)}{n} = \frac{n-i+1}{n} \)
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$E[X] = E[X_1] + \cdots + E[X_n] = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1}$
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\[E[X] = E[X_1] + \cdots + E[X_n] = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1} = n(1 + \frac{1}{2} + \cdots + \frac{1}{n}) =: nH(n)\]
Time to collect coupons

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$$
E[X] = E[X_1] + \cdots + E[X_n] = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1}
$$

$$
= n(1 + \frac{1}{2} + \cdots + \frac{1}{n}) =: nH(n) \approx n(\ln n + \gamma)
$$
Review: Harmonic sum

\[ H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \int_{1}^{n} \frac{1}{x} \, dx = \ln(n). \]
Review: Harmonic sum

\[ H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \int_1^n \frac{1}{x} \, dx = \ln(n). \]
Review: Harmonic sum

\[ H(n) = 1 \frac{1}{2} + \cdots + \frac{1}{n} \approx \int_{1}^{n} \frac{1}{x} \, dx = \ln(n). \]

A good approximation is

\[ H(n) \approx \ln(n) + \gamma \text{ where } \gamma \approx 0.58 \text{ (Euler-Mascheroni constant)}. \]
Harmonic sum: Paradox

Consider this stack of cards (no glue!):
Harmonic sum: Paradox

Consider this stack of cards (no glue!):
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Consider this stack of cards (no glue!):

If each card has length 2, the stack can extend $H(n)$ to the right of the table.
Harmonic sum: Paradox

Consider this stack of cards (no glue!):

If each card has length 2, the stack can extend $H(n)$ to the right of the table. As $n$ increases, you can go as far as you want!
Paradox

par·a·dox
/ˈperəˌdoks/

noun

a statement or proposition that, despite sound (or apparently sound) reasoning from acceptable premises, leads to a conclusion that seems senseless, logically unacceptable, or self-contradictory.
"a potentially serious conflict between quantum mechanics and the general theory of relativity known as the information paradox"

- a seemingly absurd or self-contradictory statement or proposition that when investigated or explained may prove to be well founded or true.
"in a paradox, he has discovered that stepping back from his job has increased the rewards he gleans from it"

synonyms: contradiction, contradiction in terms, self-contradiction, inconsistency, incongruity; More

- a situation, person, or thing that combines contradictory features or qualities.
"the mingling of deciduous trees with elements of desert flora forms a fascinating ecological paradox"
Stacking

Induction shows that the center of gravity after $n$ cards is $H(n)$ away from the right-most edge.

$nx = 1 - x \
\Rightarrow x = 1/(n+1)$
The cards have width 2.
The cards have width 2. Induction shows that the center of gravity after $n$ cards is $H(n)$ away from the right-most edge.
Geometric Distribution: Memoryless

Let $X$ be $G(p)$. Then, for $n \geq 0$, 

$$\Pr[X > n] = \Pr[\text{first } n \text{ flips are T}] = (1 - p)^n.$$ 

Theorem 

$$\Pr[X > n + m | X > n] = \Pr[X > m], \quad m, n \geq 0.$$ 

Proof: 

$$\Pr[X > n + m | X > n] = \Pr[X > n + m, X > n] = \Pr[X > n + m] \Pr[X > n] = (1 - p)^{n+m} = (1 - p)^m = \Pr[X > m].$$
Geometric Distribution: Memoryless

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**Theorem**

$$Pr[X > n + m | X > n] = Pr[X > m], m, n \geq 0.$$
Let $X$ be $G(p)$. Then, for $n \geq 0$,

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**Theorem**

$$Pr[X > n + m|X > n] = Pr[X > m], \ m, n \geq 0.$$  

**Proof:**

$$Pr[X > n + m|X > n] =$$
Let $X$ be $G(p)$. Then, for $n \geq 0$,

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**Theorem**

$$Pr[X > n + m | X > n] = Pr[X > m], m, n \geq 0.$$

**Proof:**

$$Pr[X > n + m | X > n] = \frac{Pr[X > n + m \text{ and } X > n]}{Pr[X > n]}$$
Geometric Distribution: Memoryless

Let $X$ be $G(\rho)$. Then, for $n \geq 0$,

$$Pr[X > n] = Pr[ \text{first } n \text{ flips are } T] = (1 - \rho)^n.$$

**Theorem**

$$Pr[X > n + m | X > n] = Pr[X > m], m, n \geq 0.$$

**Proof:**

$$Pr[X > n + m | X > n] = \frac{Pr[X > n + m \text{ and } X > n]}{Pr[X > n]}$$

$$= \frac{Pr[X > n + m]}{Pr[X > n]}$$
Geometric Distribution: Memoryless

Let $X$ be $G(p)$. Then, for $n \geq 0$,

$$Pr[X > n] = Pr[\text{first } n \text{ flips are } T] = (1 - p)^n.$$ 

**Theorem**

$$Pr[X > n + m | X > n] = Pr[X > m], m, n \geq 0.$$ 

**Proof:**

$$Pr[X > n + m | X > n] = \frac{Pr[X > n + m \text{ and } X > n]}{Pr[X > n]}$$

$$= \frac{Pr[X > n + m]}{Pr[X > n]}$$

$$= \frac{(1 - p)^{n+m}}{(1 - p)^n}$$
Geometric Distribution: Memoryless

Let $X$ be $G(p)$. Then, for $n \geq 0$,

$$Pr[X > n] = Pr[ \text{first } n \text{ flips are } T] = (1 - p)^n.$$

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$$Pr[X > n + m|X > n] = Pr[X > m], m, n \geq 0.$$ 

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$$Pr[X > n + m|X > n] = \frac{Pr[X > n + m \text{ and } X > n]}{Pr[X > n]}$$

$$= \frac{Pr[X > n + m]}{Pr[X > n]}$$

$$= \frac{(1 - p)^{n+m}}{(1 - p)^n} = (1 - p)^m$$

$$= Pr[X > m].$$
Geometric Distribution: Memoryless - Interpretation

\[ Pr[X > n + m | X > n] = Pr[X > m], m, n \geq 0. \]
Geometric Distribution: Memoryless - Interpretation

\[ Pr[X > n + m | X > n] = Pr[X > m], m, n \geq 0. \]
Geometric Distribution: Memoryless - Interpretation

\[ Pr[X > n + m | X > n] = Pr[X > m], m, n \geq 0. \]

\[ Pr[X > n + m | X > n] = Pr[A|B] = Pr[A] = Pr[X > m]. \]
Geometric Distribution: Memoryless - Interpretation

\[ Pr[X > n + m | X > n] = Pr[X > m], \quad m, n \geq 0. \]

The coin is memoryless, therefore, so is \( X \).
Theorem: For a r.v. $X$ that takes the values $\{0, 1, 2, \ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

[See later for a proof.]
Geometric Distribution: Yet another look

**Theorem:** For a r.v. $X$ that takes the values $\{0, 1, 2, \ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

[See later for a proof.]

If $X = G(p)$, then $Pr[X \geq i] = Pr[X > i - 1] = (1 - p)^{i-1}$. 
Theorem: For a r.v. $X$ that takes the values $\{0, 1, 2, \ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

[See later for a proof.]

If $X = G(p)$, then $Pr[X \geq i] = Pr[X > i - 1] = (1 - p)^{i-1}$. Hence,

$$E[X] = \sum_{i=1}^{\infty} (1 - p)^{i-1} = \sum_{i=0}^{\infty} (1 - p)^i.$$
**Theorem:** For a r.v. \(X\) that takes the values \(\{0, 1, 2, \ldots\}\), one has

\[
E[X] = \sum_{i=1}^{\infty} \Pr[X \geq i].
\]

[See later for a proof.]

If \(X = G(p)\), then \(\Pr[X \geq i] = \Pr[X > i - 1] = (1 - p)^{i-1}\).

Hence,

\[
E[X] = \sum_{i=1}^{\infty} (1 - p)^{i-1} = \sum_{i=0}^{\infty} (1 - p)^i = \frac{1}{1 - (1 - p)} =
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Theorem: For a r.v. $X$ that takes the values $\{0, 1, 2, \ldots\}$, one has

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Theorem: For a r.v. $X$ that takes values in $\{0, 1, 2, \ldots\}$, one has

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Expected Value of Integer RV

**Theorem:** For a r.v. $X$ that takes values in $\{0, 1, 2, \ldots\}$, one has

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**Proof:** One has

$$E[X] = \sum_{i=1}^{\infty} i \times Pr[X = i]$$
**Expected Value of Integer RV**

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$$E[X] = \sum_{i=1}^{\infty} i \times Pr[X = i]$$

$$= \sum_{i=1}^{\infty} i \{Pr[X \geq i] - Pr[X \geq i+1]\}. $$
Expected Value of Integer RV

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$$= \sum_{i=1}^{\infty} \{i \times Pr[X \geq i] - i \times Pr[X \geq i + 1]\}$$
Expected Value of Integer RV

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Expected Value of Integer RV

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$$= \sum_{i=1}^{\infty} Pr[X \geq i].$$
**Theorem:** For a r.v. $X$ that takes values in $\{0, 1, 2, \ldots \}$, one has

$$E[X] = \sum_{i=1}^{\infty} \Pr[X \geq i].$$

Probability mass at $i$, counted $i$ times.

\[ \ldots \quad \text{Same as } \sum_{i=1}^{\infty} i \times \Pr[X = i]. \]
Experiment: flip a coin \( n \) times. The coin is such that \( Pr[H] = \frac{\lambda}{n} \).
Random Variable: \( X \) - number of heads.
Experiment: flip a coin \( n \) times. The coin is such that \( Pr[H] = \frac{\lambda}{n} \).
Random Variable: \( X \) - number of heads. Thus, \( X = B(n, \frac{\lambda}{n}) \).
Experiment: flip a coin $n$ times. The coin is such that $Pr[H] = \lambda / n$. Random Variable: $X$ - number of heads. Thus, $X = B(n, \lambda / n)$.

**Poisson Distribution** is distribution of $X$ “for large $n$.”
Poisson Experiment: flip a coin \( n \) times. The coin is such that \( Pr[H] = \lambda / n \).

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Experiment: flip a coin $n$ times. The coin is such that $Pr[H] = \lambda / n$. Random Variable: $X$ - number of heads. Thus, $X = B(n, \lambda / n)$. **Poisson Distribution** is distribution of $X$ “for large $n$.” We expect $X \ll n$. 
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**Poisson Distribution** is distribution of $X$ “for large $n$.”
We expect $X \ll n$. For $m \ll n$ one has

$$Pr[X = m] = \left(\frac{n}{m}\right) p^m (1 - p)^{n-m},$$
where $p = \lambda/n = n\left(n - 1\right) \cdots \left(n - m + 1\right) / m!$$
$$\approx \left(\frac{1}{2}\right) \lambda^m m! \left(1 - \lambda n\right)^{n-m} \approx \left(\frac{1}{2}\right) \lambda^m m! e^{-\lambda}.$$
Poisson

Experiment: flip a coin \( n \) times. The coin is such that \( Pr[H] = \lambda / n \).
Random Variable: \( X \) - number of heads. Thus, \( X = B(n, \lambda / n) \).
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Pr[X = m] = \binom{n}{m} p^m (1 - p)^{n-m}, \text{ with } p =
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Experiment: flip a coin $n$ times. The coin is such that $Pr[H] = \lambda / n$. Random Variable: $X$ - number of heads. Thus, $X = B(n, \lambda / n)$. **Poisson Distribution** is distribution of $X$ “for large $n$.” We expect $X \ll n$. For $m \ll n$ one has

$$Pr[X = m] = \binom{n}{m} p^m (1 - p)^{n-m}, \text{ with } p = \lambda / n$$

$$= \frac{n(n-1) \cdots (n-m+1)}{m!} \left(\frac{\lambda}{n}\right)^m \left(1 - \frac{\lambda}{n}\right)^{n-m}$$
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Experiment: flip a coin $n$ times. The coin is such that $Pr[H] = \lambda / n$.

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**Definition** Poisson Distribution with parameter $\lambda > 0$

$$X = P(\lambda) \iff Pr[X = m] = \frac{\lambda^m}{m!} e^{-\lambda}, m \geq 0.$$
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Simeon Poisson

The Poisson distribution is named after:
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The Poisson distribution is named after:
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Equal Time: B. Geometric

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Prof. Walrand could not find a picture of D. Binomial,
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Review: Distributions
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Obvious.
Independence: Examples

**Example 1**
Roll two die. \( X, Y = \) number of pips on the two dice. \( X, Y \) are independent.

Indeed:
\[
\Pr[X = a, Y = b] = \frac{1}{36}, \quad \Pr[X = a] = \Pr[Y = b] = \frac{1}{6}.
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**Example 2**
Roll two die. \( X = \) total number of pips, \( Y = \) number of pips on die 1 minus number on die 2. \( X \) and \( Y \) are not independent.

Indeed:
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\Pr[X = 12, Y = 1] = 0 \neq \Pr[X = 12] \Pr[Y = 1] > 0.
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Flip a fair coin five times, $X$ = number of Hs in first three flips, $Y$ = number of Hs in last two flips. $X$ and $Y$ are independent.

Indeed: $Pr[X = a, Y = b] = \left(\frac{3}{2}a\right)\left(\frac{2}{5}b\right)\left(\frac{3}{5}a\right)\left(\frac{2}{5}b\right)\left(\frac{3}{5}a\right) = Pr[X = a] Pr[Y = b] = \frac{1}{6}$. 

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**Example 1**
Roll two die. \( X, Y = \) number of pips on the two dice. \( X, Y \) are independent.

Indeed: \[ Pr[X = a, Y = b] = \frac{1}{36}, Pr[X = a] = Pr[Y = b] = \frac{1}{6}. \]

**Example 2**
Roll two die. \( X = \) total number of pips, \( Y = \) number of pips on die 1 minus number on die 2. \( X \) and \( Y \) are not independent.

Indeed: \[ Pr[X = 12, Y = 1] = 0 \neq Pr[X = 12]Pr[Y = 1] > 0. \]

**Example 3**
Flip a fair coin five times, \( X = \) number of \( H \)s in first three flips, \( Y = \) number of \( H \)s in last two flips. \( X \) and \( Y \) are independent.

Indeed:

\[
Pr[X = a, Y = b] = \binom{3}{a} \binom{2}{b} 2^{-5} = \binom{3}{a} 2^{-3} \times \binom{2}{b} 2^{-2} = Pr[X = a]Pr[Y = b].
\]
A useful observation about independence

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**Theorem**

$X$ and $Y$ are independent if and only if

$$Pr[X \in A, Y \in B] = Pr[X \in A]Pr[Y \in B]$$

for all $A, B \subset \mathbb{R}$. 

Proof:

If $(\Leftarrow)$: Choose $A = \{a\}$ and $B = \{b\}$. This shows that

$$Pr[X = a, Y = b] = Pr[X = a]Pr[Y = b].$$

Only if $(\Rightarrow)$:

$$Pr[X \in A, Y \in B] = \sum_{a \in A} \sum_{b \in B} Pr[X = a, Y = b] = \sum_{a \in A} \sum_{b \in B} Pr[X = a]Pr[Y = b] = \sum_{a \in A} \left( \sum_{b \in B} Pr[Y = b] \right) Pr[X = a] = \sum_{a \in A} Pr[X = a]Pr[Y \in B] = \cdots.$$
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Let $X, Y$ be independent RV. Then

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Functions of Independent random Variables

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\[\square\]
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$$E[(X + 2Y + 3Z)^2] = E[X^2 + 4Y^2 + 9Z^2 + 4XY + 12YZ + 6XZ]$$
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$$= \frac{1 + 3n + 2n^2}{3} - \frac{(n+1)^2}{2}.$$
Mutually Independent Random Variables

Definition
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$X, Y, Z$ are mutually independent if

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for all $x, y, z$. 
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The events $A, B, C, \ldots$ are pairwise (resp. mutually) independent iff the random variables $1_A, 1_B, 1_C, \ldots$ are pairwise (resp. mutually) independent.
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The events $A, B, C, \ldots$ are pairwise (resp. mutually) independent iff the random variables $1_A, 1_B, 1_C, \ldots$ are pairwise (resp. mutually) independent.

Proof:

$$Pr[1_A = 1, 1_B = 1, 1_C = 1] = Pr[A \cap B \cap C], \ldots$$
Functions of pairwise independent RVs

If $X, Y, Z$ are pairwise independent, but not mutually independent, it may be that

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If $X, Y, Z$ are pairwise independent, but not mutually independent, it may be that $f(X)$ and $g(Y, Z)$ are not independent.

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Functions of mutually independent RVs

Theorem
Functions of disjoint collections of mutually independent random variables are mutually independent.

Example:
Let \( \{X_n, n \geq 1\} \) be mutually independent.

Then,

\[
Y_1 := X_1 X_2 (X_3 + X_4)^2,
Y_2 := \max\{X_5, X_6\} - \min\{X_7, X_8\},
Y_3 := X_9 \cos(X_{10} + X_{11})
\]

are mutually independent.

Proof:
Let \( B_1 := \{ (x_1, x_2, x_3, x_4) | x_1 x_2 (x_3 + x_4)^2 \in A_1 \} \). Similarly for \( B_2, B_2 \).

Then

\[
\Pr[Y_1 \in A_1, Y_2 \in A_2, Y_3 \in A_3] = \Pr[(X_1, \ldots, X_4) \in B_1] \Pr[(X_5, \ldots, X_8) \in B_2] \Pr[(X_9, \ldots, X_{11}) \in B_3]
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Functions of disjoint collections of mutually independent random variables are mutually independent.

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Operations on Mutually Independent Events

Theorem

Operations on disjoint collections of mutually independent events produce mutually independent events. For instance, if $A$, $B$, $C$, $D$, $E$ are mutually independent, then $A \Delta B$, $C \setminus D$, $\bar{E}$ are mutually independent.

Proof:

$1 \ A \Delta B = f(1_A, 1_B)$ where $f(0, 0) = 0$, $f(1, 0) = 1$, $f(0, 1) = 1$, $f(1, 1) = 0$.

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Theorem

Let $X_1, \ldots, X_n$ be mutually independent RVs. Then,

$$E[X_1X_2\cdots X_n] = E[X_1]E[X_2]\cdots E[X_n].$$

Proof:

Assume that the result is true for $n$. (It is true for $n=2$.) Then, with $Y = X_1\cdots X_n$, one has

$$E[X_1\cdots X_nX_{n+1}] = E[XY_{n+1}],$$

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Distributions; Independence
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Distributions:

▶ \(G(p)\) : \(E[X] = \frac{1}{p}\);

▶ \(B(n, p)\) : \(E[X] = np\);

▶ \(P(\lambda)\) : \(E[X] = \lambda\)

Independence:

▶ \(X, Y\) independent \(\iff\) \(\Pr[X \in A, Y \in B] = \Pr[X \in A] \cdot \Pr[Y \in B]\)

▶ Then, \(f(X), g(Y)\) are independent and \(E[XY] = E[X] \cdot E[Y]\)

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