CS70: Lecture 21.

Variance; Inequalities; WLLN

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1. Review: Distributions
2. Review: Independence
3. Variance
4. Inequalities
   - Markov
   - Chebyshev
5. Weak Law of Large Numbers

Review: Distributions

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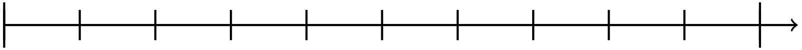

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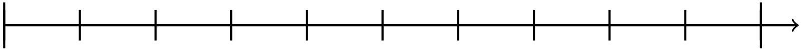

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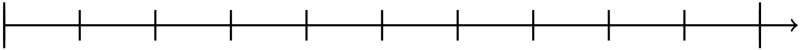

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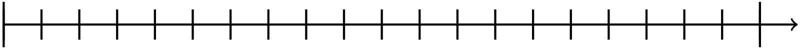

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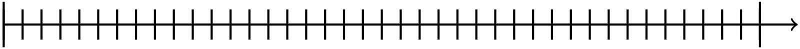

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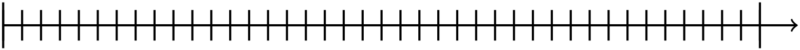

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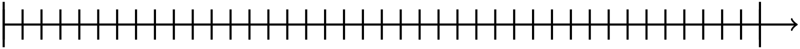

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.... $Pr[X = i] = \binom{n}{i} p^i (1-p)^{n-i}$.
derive simple expression.

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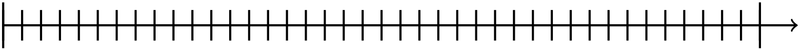

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...And we get the Poisson distribution!

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Random Variable: X - number of heads.

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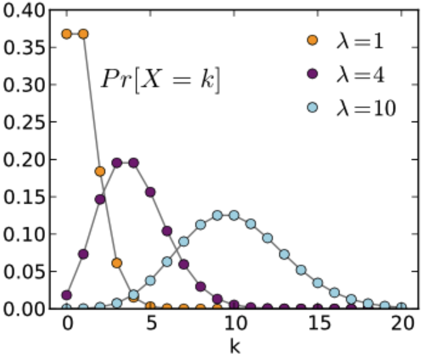

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For (1) we used $m \ll n$; for (2) we used $(1 - a/n)^n \approx e^{-a}$.

Simeon Poisson

The Poisson distribution is named after:

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The Poisson distribution is named after:

Siméon Poisson

Siméon Denis Poisson (1781–1840)

“Life is good for only two things: doing mathematics and teaching it.”

Review: Independence

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$\Rightarrow E[XY] = E[X]E[Y].$

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$X, Y, Z, V, W, U \ldots$ are mutually independent

$\Rightarrow f(X, Y), g(Z, V, W), h(U, \ldots), \ldots$ are mutually independent

Review: Mutual Independence

Definition

$X, Y, Z, \ldots$ are mutually independent

$$\Leftrightarrow Pr[X = x, Y = y, Z = z, \ldots]$$

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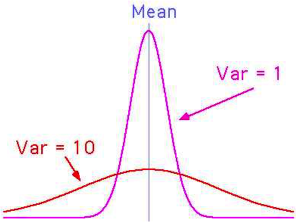

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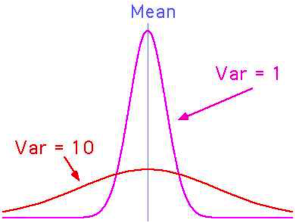

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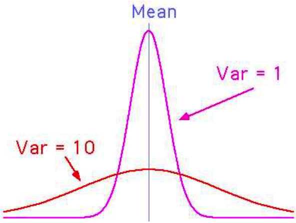

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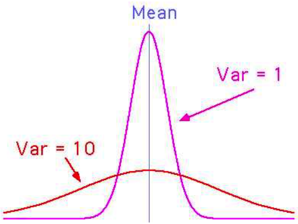

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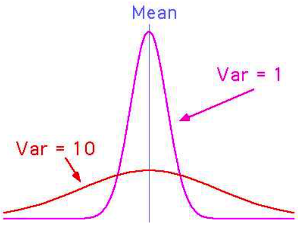

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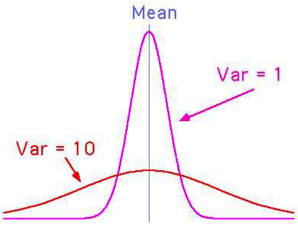

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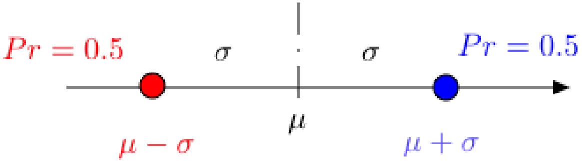

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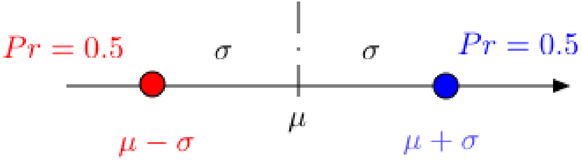

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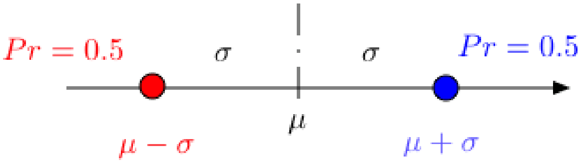

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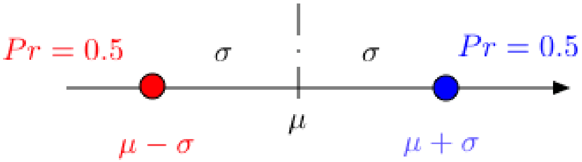

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Exercise: How big can you make $\frac{\sigma(X)}{E[|X - E[X]|]}$?

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This gives

$$var(X) = \frac{1+3n+2n^2}{6} - \frac{(n+1)^2}{4} = \frac{n^2-1}{12}.$$

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&= 2(\frac{1}{p})-1 = \frac{2-p}{p}
\end{aligned}$$

$\implies E[X^2]=(2-p)/p^2$ and
$var[X]=E[X^2]-E[X]^2=\frac{2-p}{p^2}-\frac{1}{p^2}=\frac{1-p}{p^2}$.
$\sigma(X)=\frac{\sqrt{1-p}}{p}\approx E[X]$ when p is small(ish).

Variance of geometric distribution.

X is a geometrically distributed RV with parameter p.
Thus, $Pr[X = n] = (1-p)^{n-1}p$ for $n \geq 1$. Recall $E[X] = 1/p$.

$$
\begin{aligned}
E[X^2] &= p + 4p(1-p) + 9p(1-p)^2 + ... \\
-(1-p)E[X^2] &= \quad -[p(1-p) + 4p(1-p)^2 + ...] \\
pE[X^2] &= p + 3p(1-p) + 5p(1-p)^2 + ... \\
&= 2(p + 2p(1-p) + 3p(1-p)^2 + ..) \quad E[X]! \\
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Maybe not much easier...but there is a payoff.

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□

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That is, we assume that $E[X] = E[Y] = \cdots = 0$.

Then, by independence,

$$E[XY] = E[X]E[Y] = 0. \text{ Also, } E[XZ] = E[YZ] = \cdots = 0.$$

Hence,

$$\begin{aligned} var(X + Y + Z + \cdots) &= E((X + Y + Z + \cdots)^2) \\ &= E(X^2 + Y^2 + Z^2 + \cdots + 2XY + 2XZ + 2YZ + \cdots) \\ &= E(X^2) + E(Y^2) + E(Z^2) + \cdots + 0 + \cdots + 0 \\ &= var(X) + var(Y) + var(Z) + \cdots . \end{aligned}$$

□

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Flip coin with heads probability p.

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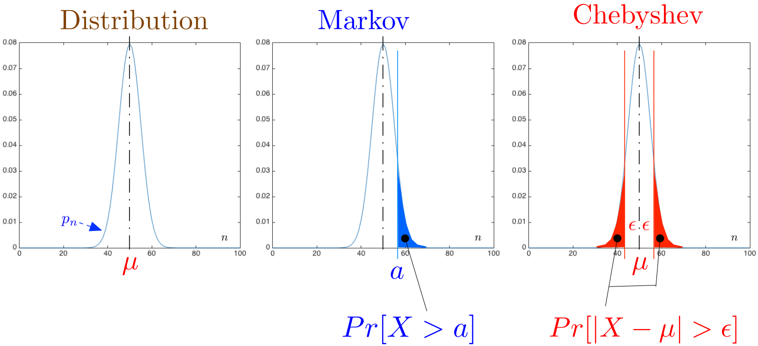
Inequalities: An Overview



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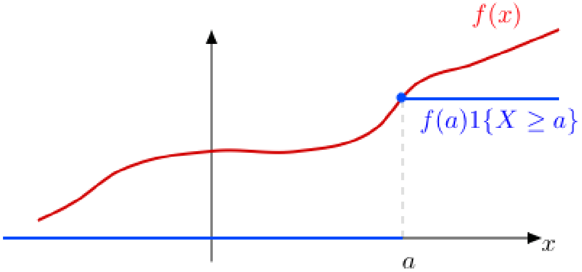
Taking the expectation yields the inequality, because expectation is monotone. $\square$

A picture



$$f(a)1\{X \geq a\} \leq f(x) \Rightarrow 1\{X \geq a\} \leq \frac{f(X)}{f(a)}$$

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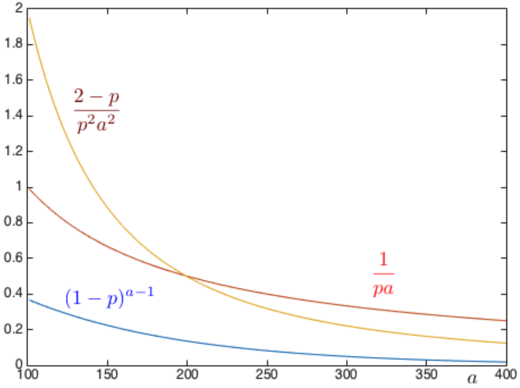

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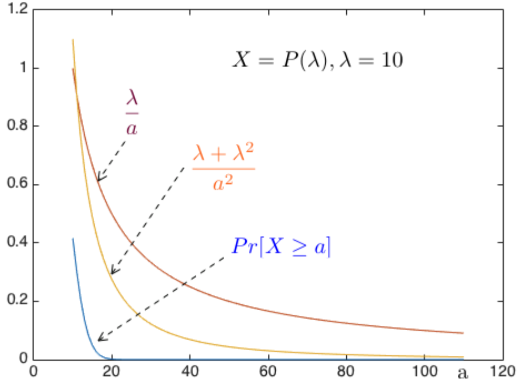

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This result confirms that the variance measures the "deviations from the mean."

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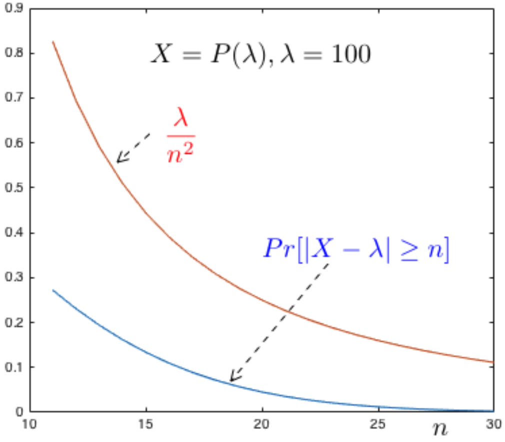

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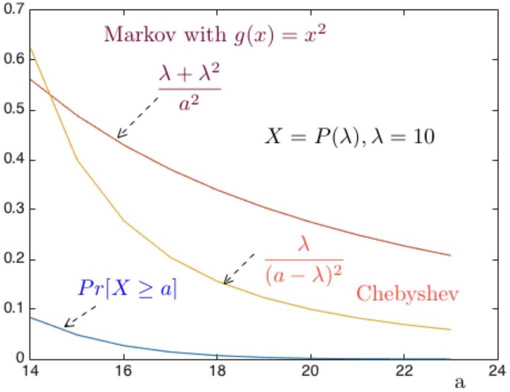

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We look at a general case next.

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