Markov Chains

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Two-State Markov Chain

Here is a symmetric two-state Markov chain. It describes a random motion in \{0, 1\}. Here, \(a\) is the probability that the state changes in the next step.

Let’s simulate the Markov chain:

\[
\begin{align*}
&\begin{array}{c}
1 - a \\
0 \\
1 - a
\end{array} \\
&\begin{array}{c}
a \\
0 \\
a
\end{array} \\
&\begin{array}{c}
a \\
1 \\
a
\end{array}
\end{align*}
\]

\(a = 0.1\)

\(a = 0.2\)

\(a = 0.4\)
Five-State Markov Chain

At each step, the MC follows one of the outgoing arrows of the current state, with equal probabilities.

Let's simulate the Markov chain:
Finite Markov Chain: Definition

- A finite set of states: $\mathcal{X} = \{1, 2, \ldots, K\}$
- A probability distribution $\pi_0$ on $\mathcal{X}$: $\pi_0(i) \geq 0, \sum_i \pi_0(i) = 1$
- Transition probabilities: $P(i,j)$ for $i,j \in \mathcal{X}$
  
  \[ P(i,j) \geq 0, \forall i,j; \sum_j P(i,j) = 1, \forall i \]

- $\{X_n, n \geq 0\}$ is defined so that

  \[ Pr[X_0 = i] = \pi_0(i), i \in \mathcal{X} \text{ (initial distribution)} \]

  \[ Pr[X_{n+1} = j \mid X_0, \ldots, X_n = i] = P(i,j), i,j \in \mathcal{X}. \]
First Passage Time - Example 1

Let's flip a coin with $Pr[H] = p$ until we get $H$. How many flips, on average?

Let's define a Markov chain:

- $X_0 = S$ (start)
- $X_n = S$ for $n \geq 1$, if last flip was $T$ and no $H$ yet
- $X_n = E$ for $n \geq 1$, if we already got $H$ (end)
First Passage Time - Example 1

Let’s flip a coin with $Pr[H] = p$ until we get $H$. How many flips, on average?

Let $\beta(S)$ be the average time until $E$, starting from $S$. Then,

$$\beta(S) = 1 + q\beta(S) + p0.$$ 

(See next slide.) Hence,

$$p\beta(S) = 1,$$

so that $\beta(S) = 1/p$.

Note: Time until $E$ is $G(p)$. We have rediscovered that the mean of $G(p)$ is $1/p$. 
Let’s flip a coin with $Pr[H] = p$ until we get $H$. How many flips, on average?

Let $\beta(S)$ be the average time until $E$. Then,

$$\beta(S) = 1 + q\beta(S) + p0.$$ 

**Justification:** Let $N$ be the random number of steps until $E$, starting from $S$. Let also $N'$ be the number of steps until $E$, after the second visit to $S$. Finally, let $Z = 1\{\text{first flip } = H\}$. Then,

$$N = 1 + (1 - Z) \times N' + Z \times 0.$$ 

Now, $Z$ and $N'$ are independent. Also, $E[N'] = E[N] = \beta(S)$. Hence, taking expectation,

$$\beta(S) = E[N] = 1 + (1 - p)E[N'] + p0 = 1 + q\beta(S) + p0.$$
First Passage Time - Example 2

Let’s flip a coin with $Pr[H] = p$ until we get two consecutive $H$s. How many flips, on average?

$$H \, T \, H \, T \, T \, T \, H \, T \, H \, T \, H \, T \, H \, T \, H \, H$$

Let’s define a Markov chain:

- $X_0 = S$ (start)
- $X_n = E$, if we already got two consecutive $H$s (end)
- $X_n = T$, if last flip was $T$ and we are not done
- $X_n = H$, if last flip was $H$ and we are not done
First Passage Time - Example 2

Let's flip a coin with $Pr[H] = p$ until we get two consecutive $H$s. How many flips, on average? Here is a picture:

Let $\beta(i)$ be the average time from state $i$ until the MC hits state $E$.

We claim that (these are called the first step equations)

$$\beta(S) = 1 + p\beta(H) + q\beta(T)$$
$$\beta(H) = 1 + p0 + q\beta(T)$$
$$\beta(T) = 1 + p\beta(H) + q\beta(T).$$

Solving, we find $\beta(S) = 2 + 3qp^{-1} + q^2p^{-2}$. (E.g., $\beta(S) = 6$ if $p = 1/2$.)
Let us justify the first step equation for $\beta(T)$. The others are similar.

Let $N(T)$ be the random number of steps, starting from $T$ until the MC hits $E$. Let also $N(H)$ be defined similarly. Finally, let $N'(T)$ be the number of steps after the second visit to $T$ until the MC hits $E$. Then,

$$N(T) = 1 + Z \times N(H) + (1 - Z) \times N'(T)$$

where $Z = 1\{\text{first flip in } T \text{ is } H\}$. Since $Z$ and $N(H)$ are independent, and $Z$ and $N'(T)$ are independent, taking expectations, we get

$$E[N(T)] = 1 + pE[N(H)] + qE[N'(T)],$$

i.e.,

$$\beta(T) = 1 + p\beta(H) + q\beta(T).$$
First Passage Time - Example 3

You roll a balanced six-sided die until the sum of the last two rolls is 8. How many times do you have to roll the die, on average?

\[ \beta(S) = 1 + \frac{1}{6} \sum_{j=1}^{6} \beta(j) ; \beta(1) = 1 + \frac{1}{6} \sum_{j=1}^{6} \beta(j) ; \beta(i) = 1 + \frac{1}{6} \sum_{j=1,...,6;j \neq 8-i} \beta(j) , i = 2, \ldots , 6. \]

Symmetry: \( \beta(2) = \cdots = \beta(6) =: \gamma \). Also, \( \beta(1) = \beta(S) \). Thus,

\[ \beta(S) = 1 + (5/6)\gamma + \beta(S)/6; \quad \gamma = 1 + (4/6)\gamma + (1/6)\beta(S). \]

\[ \Rightarrow \cdots \beta(S) = 8.4. \]
You try to go up a ladder that has 20 rungs. At each time step, you succeed in going up by one rung with probability $p = 0.9$. Otherwise, you fall back to the ground. How many time steps does it take you to reach the top of the ladder, on average?

$$
\beta(n) = 1 + p\beta(n+1) + q\beta(0), \quad 0 \leq n < 19
$$

$$
\beta(19) = 1 + p0 + q\beta(0)
$$

$$
\Rightarrow \beta(0) = \frac{p^{-20} - 1}{1 - p} \approx 72.
$$

See Lecture Note 24 for algebra.
You play a game of “heads or tails” using a biased coin that yields ‘heads’ with probability $p < 0.5$. You start with $10$. At each step, if the flip yields ‘heads’, you earn $1$. Otherwise, you lose $1$. What is the probability that you reach $100$ before $0$?

Let $\alpha(n)$ be the probability of reaching $100$ before $0$, starting from $n$, for $n = 0, 1, \ldots, 100$.

\[ \alpha(0) = 0; \alpha(100) = 1. \]

\[ \alpha(n) = p\alpha(n+1) + q\alpha(n-1), 0 < n < 100. \]

\[ \Rightarrow \alpha(n) = \frac{1 - \rho^n}{1 - \rho^{100}} \text{ with } \rho = qp^{-1}. \text{ (See LN 24)} \]
You play a game of “heads or tails” using a biased coin that yields ‘heads’ with probability 0.48. You start with $10. At each step, if the flip yields ‘heads’, you earn $1. Otherwise, you lose $1. What is the probability that you reach $100 before $0?

Morale of example: Be careful!
Summary: First Step Equations

Let $X_n$ be a MC on $\mathcal{X}$ and $A, B \subset \mathcal{X}$ with $A \cap B = \emptyset$. Define

$$T_A = \min\{ n \geq 0 \mid X_n \in A \} \text{ and } T_B = \min\{ n \geq 0 \mid X_n \in B \}.$$ 

Let

$$\beta(i) = E[T_A \mid X_0 = i] \text{ and } \alpha(i) = Pr[T_A < T_B \mid X_0 = i], i \in \mathcal{X}.$$ 

The FSE are

$$\beta(i) = \begin{cases} 0, & i \in A \\ 1 + \sum_{j} P(i,j) \beta(j), & i \notin A \end{cases}$$

$$\alpha(i) = \begin{cases} 1, & i \in A \\ 0, & i \in B \\ \sum_{j} P(i,j) \alpha(j), & i \notin A \cup B. \end{cases}$$