CS70: Jean Walrand: Lecture 29.

Probability Review

1. True or False
2. Some Key Results
3. Quiz 1: G ($\approx 40\%$)
4. Quiz 2: PG ($\approx 40\%$)
5. Quiz 3: R ($\approx 20\%$)
6. Common Mistakes

True or False

- Ω and A are independent. True
- $Pr[A \cap B] = Pr[A] + Pr[B] - Pr[A \cup B]$. True
- $Pr[A \setminus B] \geq Pr[A] - Pr[B]$. True
- $X_1, \ldots, X_n$ i.i.d. $\implies var(\frac{X_1 + \cdots + X_n}{n}) = var(X_1)$. False: $\times \frac{1}{n}$
- $Pr[|X - a| \geq b] \leq \frac{E[(X-a)^2]}{b^2}$. True
- $X_1, \ldots, X_n$ i.i.d. $\implies \frac{X_1 + \cdots + X_n - nE[X_1]}{n\sigma(X_1)} \to \mathcal{N}(0,1)$. False: $\sqrt{n}$
- $X = Expo(\lambda) \implies Pr[X > 5 | X > 3] = Pr[X > 2]$. True:
  $\frac{\exp\{-\lambda 5\}}{\exp\{-\lambda 3\}} = \exp\{-\lambda 2\}$.

Correct or not?

When $n \gg 1$, one has

- $[A_n - 2\sigma\frac{1}{n}, A_n + 2\sigma\frac{1}{n}] = 95\%$-CI for μ. No
- $[A_n - 2\sigma\frac{1}{\sqrt{n}}, A_n + 2\sigma\frac{1}{\sqrt{n}}] = 95\%$-CI for μ. Yes
- If $0.3 < \sigma < 3$, then
  $[A_n - 0.6\frac{1}{\sqrt{n}}, A_n + 0.6\frac{1}{\sqrt{n}}] = 95\%$-CI for μ. No
- If $0.3 < \sigma < 3$, then
  $[A_n - 6\frac{1}{\sqrt{n}}, A_n + 6\frac{1}{\sqrt{n}}] = 95\%$-CI for μ. Yes

Match Items

[1] $Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}$;

[2] $Pr[|X - E[X]| > a] \leq \frac{var[X]}{a^2}$

[3] $Pr[X \geq a] \leq \min_{\theta > 0} \frac{E[e^{\theta X}]}{e^{\theta a}}$

[4] $g(\cdot)$ convex $\Rightarrow E[g(X)] \geq g(E[X])$

[5] $E[Y] + \frac{cov(X, Y)}{var(X)}(X - E[X])$.

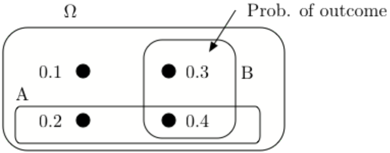
[6] $\sum_y y Pr[Y = y | X = x]$

[7] $Pr[|\frac{X_1 + \cdots + X_n}{n} - E[X_1]| \geq \epsilon] \to 0$,

[8] $E[(Y - E[Y|X])h(X)] = 0$.

- WLLN (7)
- MMSE (6)
- Projection property (8)
- Chebyshev (2)
- LLSE (5)
- Markov's inequality (1)

Quiz 1: G



1. What is $P[A|B]$?

   $Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} = \frac{0.4}{0.7}$

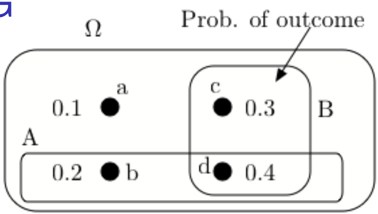
2. What is $Pr[B|A]$?

   $Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]} = \frac{0.4}{0.6}$

3. Are A and B positively correlated?

   No. $Pr[A \cap B] = 0.4 < Pr[A]Pr[B] = 0.6 \times 0.7$.

Quiz 1: G



ω	$X(\omega)$	$Y(\omega)$
a	0	0
b	1	0
c	0	2
d	1	2

4. What is $E[Y|X]$?

$$\begin{aligned} E[Y|X=0] &= 0 \times Pr[Y=0|X=0] + 2 \times Pr[Y=2|X=0] \\ &= 2 \times \frac{0.3}{0.4} = 1.5 \\ E[Y|X=1] &= 0 \times Pr[Y=0|X=1] + 2 \times Pr[Y=2|X=1] \\ &= 2 \times \frac{0.4}{0.6} = 1.33 \end{aligned}$$

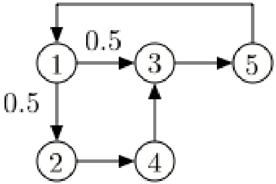
5. What is $cov(X, Y)$?

   $cov(X, Y) = E[XY] - E[X]E[Y] = 0.8 - 0.6 \times 1.4 = -0.04$

6. What is $L[Y|X]$?

   $L[Y|X] = E[Y] + \frac{cov(X,Y)}{var(X)}(X - E[X]) = 1.4 + \frac{-0.04}{0.6 \times 0.4}(X - 0.6)$

Quiz 1: G

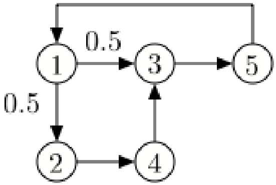


7. Is this Markov chains irreducible? Yes.
8. Is this Markov chain periodic?

   No. The return times to 3 are $\{3, 5, ..\}$: coprime!
9. Does π_n converge to a value independent of π_0? Yes!
10. Does $\frac{1}{n}\sum_{m=1}^{n-1} 1\{X_m = 3\}$ converge as $n \to \infty$? Yes!
11. Calculate π.

    Let $a = \pi(1)$. Then $a = \pi(5), \pi(2) = 0.5a, \pi(4) = \pi(2) = 0.5a, \pi(3) = 0.5\pi(1) + \pi(4) = a$. Thus,
    $\pi = [a, 0.5a, a, 0.5a, a] = [1, 0.5, 1, 0.5, 1]a$, so $a = 1/4$.

Quiz 1: G



12. Write the first step equations for calculating the mean time from 1 to 4.

$$\beta(1) = 1 + 0.5\beta(2) + 0.5\beta(3)$$
$$\beta(2) = 1$$
$$\beta(3) = 1 + \beta(5)$$
$$\beta(5) = 1 + \beta(1).$$

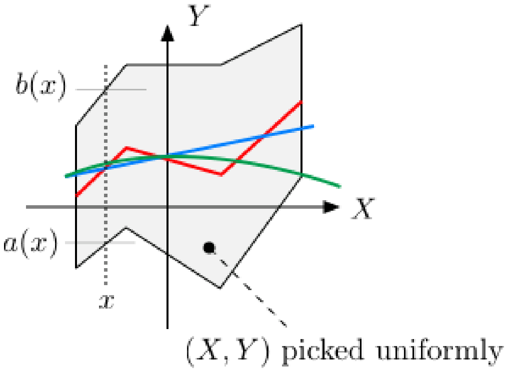
13. Solve these equations.

$$\begin{aligned} \beta(1) &= 1 + 0.5 \times 1 + 0.5 \times (1 + (1 + \beta(1))) \\ &= 2.5 + 0.5\beta(1). \end{aligned}$$

Hence, $\beta(1) = 5$.

Quiz 1: G

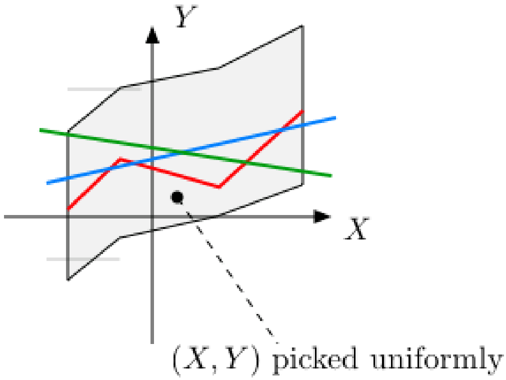
14. Which is $E[Y|X]$? Blue, red or green?



Answer: Red.
Given $X = x$, $Y = U[a(x), b(x)]$. Thus, $E[Y|X = x] = \frac{a(x)+b(x)}{2}$.

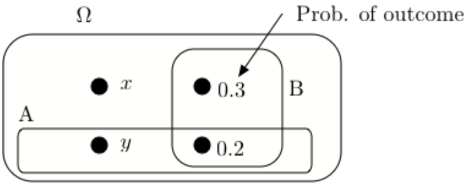
Quiz 1: G

15. Which is $L[Y|X]$? Blue, red or green?



Answer: Blue.
Cannot be red (not a straight line).
Cannot be green: X and Y are clearly positively correlated.

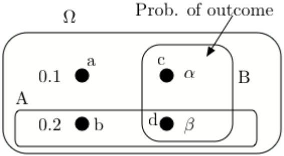
Quiz 2: PG



1. Find (x, y) so that A and B are independent.
   We need
   $$Pr[A \cap B] = Pr[A]Pr[B]$$
   That is,
   $$0.2 = (y + 0.2) \times 0.5 = 0.5y + 0.1$$
   Hence,
   $$y = 0.2 \text{ and } x = 0.3.$$

2. Find the value of x that maximizes $Pr[B|A]$.
   Obviously, it is $x = 0.5$.

Quiz 2: PG



ω	$X(\omega)$	$Y(\omega)$
a	0	0
b	1	0
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d	1	2

3. Find α so that X and Y are independent.

   We need

$$Pr[X = 0, Y = 0] = Pr[X = 0]Pr[Y = 0]$$

   That is,

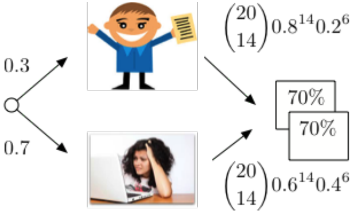
$$0.1 = (0.1 + \alpha) \times (0.1 + 0.2) = 0.03 + 0.3\alpha$$

   Hence,

$$\alpha = 0.233$$

Quiz 2: PG

4. A CS70 student is great w.p. 0.3 and good w.p. 0.7. A great student solves each question correctly w.p. 0.8 whereas a good student does it w.p. 0.6. One student got right 70% of the 10 questions on Midterm 1 and 70% of the 10 questions on Midterm 2. What is the expected score of the student on the final?



$$p := Pr[\text{great}|\text{scores}] = \frac{0.3\binom{20}{14}0.8^{14}0.2^{6}}{0.3\binom{20}{14}0.8^{14}0.2^{6} + 0.7\binom{20}{14}0.6^{14}0.4^{6}}$$

$$= \frac{(0.3)0.8^{14}0.2^{6}}{(0.3)0.8^{14}0.2^{6} + (0.7)0.6^{14}0.4^{6}} \approx 0.27$$

Expected score $= p80\% + (1-p)60\% \approx 65\%.$

Quiz 2: PG

5. You roll a balanced six-sided die 20 times. Use CLT to upper-bound the probability that the total number of dots exceeds 85.

Let $X = X_1 + \cdots + X_{20}$ be the total number of dots.
Then
$$\frac{X - 70}{\sigma\sqrt{20}} \approx \mathcal{N}(0,1)$$
where
$$\sigma^2 = var(X_1) = (1/6)\sum_{m=1}^{6} m^2 - (3.5)^2 \approx 2.9 = 1.7^2.$$
Now,
$$\begin{aligned} Pr[X > 85] &= Pr[X - 70 > 15] \\ &= Pr[\frac{X-70}{1.7 \times 4.5} > \frac{15}{1.7 \times 4.5}] \\ &= Pr[\frac{X-70}{1.7 \times 4.5} > 2] \approx 2.5\%. \end{aligned}$$

Quiz 2: PG

6. You roll a balanced six-sided die 20 times. Use Chebyshev to upper-bound the probability that the total number of dots exceeds 85.

Let $X = X_1 + \cdots + X_{20}$ be the total number of dots.
Then

$$\begin{aligned} Pr[X > 85] &= Pr[X - 70 > 15] \leq Pr[|X - 70| > 15] \\ &\leq \frac{var(X)}{15^2}. \end{aligned}$$

Now,

$$var(X) = 20 var(X_1) = 20 \times 2.9 = 58.$$

Hence,

$$Pr[X > 85] \leq \frac{58}{15^2} \approx 0.26.$$

Quiz 2: PG

7. Let X, Y, Z be i.i.d. $Expo(1)$. Find $L[X|X+2Y+3Z]$.

   Let $V = X + 2Y + 3Z$. One finds

$$L[X|V] = E[X] + \frac{cov(X,V)}{var(V)}(V - E[V])$$
$$E[X] = 1, E[V] = 6$$
$$cov(X,V) = var(X) = 1$$
$$var(V) = 1 + 4 + 9 = 14.$$

   Hence,

$$L[X|V] = 1 + \frac{1}{14}(V - 6).$$

8. Let X, Y, Z be i.i.d. $Expo(1)$. Calculate $E[X+Z|X+Y]$.

$$\begin{aligned} E[X+Z|X+Y] &= E[X|X+Y] + E[Z] \\ &= \frac{1}{2}(X+Y) + 1. \end{aligned}$$

9. Let X, Y, Z be i.i.d. $Expo(1)$. Calculate $L[X+Z|X+Y]$.

$$L[X+Z|X+Y] = \frac{1}{2}(X+Y) + 1.$$

Q2: PG

10. You roll a balanced die.

    You start with \$1.00.

    Every time you get a 6, your fortune is multiplied by 10.

    Every time you do not get a 6, your fortune is divided by 2.

    Let X_n be your fortune at the start of step n,

    Calculate $E[X_n]$.

We have $X_1 = 1$. Also,

$$\begin{aligned} E[X_{n+1}|X_n] &= X_n \times [10\frac{1}{6} + 0.5 \times \frac{5}{6}] \\ &= \rho X_n, \rho = 10\frac{1}{6} + 0.5 \times \frac{5}{6} \approx 2.1. \end{aligned}$$

Hence,

$$E[X_{n+1}] = \rho E[X_n], n \geq 1.$$

Thus,

$$E[X_n] = \rho^{n-1}, n \geq 1.$$

Quiz 3: R

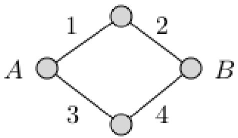
1. The lifespans of good lightbulbs are exponentially distributed with mean 1 year. Those of defective bulbs are exponentially distributed with mean 0.8. All the bulbs in one batch are equally likely to be good or defective. You test one bulb and note that it burns out after 0.6 year. (a) What is the probability you got a batch of good bulbs? (b) What is the expected lifespan of another bulb in that batch?

   Hint: If $X = Expo(\lambda), f_X(x) = \lambda e^{-\lambda x} 1\{x > 0\}, E[X] = 1/\lambda$.

Let X be the lifespan of a bulb, G the event that it is good, and B the event that it is bad.

$$
\begin{aligned}
(a) \quad & p := Pr[G|X \in (0.6, 0.6+\delta)] \\
& = \frac{0.5 Pr[X \in (0.6, 0.6+\delta)|G]}{0.5 Pr[X \in (0.6, 0.6+\delta)|G] + 0.5 Pr[X \in (0.6, 0.6+\delta)|D]} \\
& = \frac{e^{-0.6}\delta}{e^{-0.6}\delta + (0.8)^{-1} e^{-(0.8)^{-1} 0.6}\delta} \approx 0.488. \\
(b) \quad & E[\text{ lifespan of other bulb }] = p \times 1 + (1-p) \times 0.8 \approx 0.9.
\end{aligned}
$$

Quiz 3: R



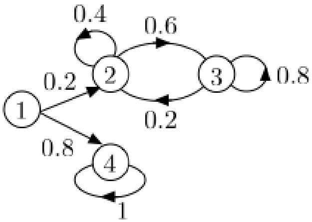
2. In the figure, $1, 2, 3, 4$ are links that fail after i.i.d. times that are $U[0,1]$.

   Find the average time until A and B are disconnected.

Let X_k be the lifespan of link k, for $k = 1, \ldots, 4$.
We are looking for $E[Z]$ where $Z = \max\{Y_1, Y_2\}$ with $Y_1 = \min\{X_1, X_2\}$ and $Y_2 = \min\{X_3, X_4\}$.

$$
\begin{aligned}
Pr[Y_1 > t] &= Pr[X_1 > t]Pr[X_2 > t] = (1-t)^2 \\
Pr[Z \leq t] &= Pr[Y_1 \leq t]Pr[Y_2 \leq t] = (1-(1-t)^2)^2 \\
&= (2t - t^2)^2 = 4t^2 - 4t^3 + t^4 \\
f_Z(t) &= 8t - 12t^2 + 4t^3 \\
E[Z] &= \int_0^1 t f_Z(t) dt = 8\frac{1}{3} - 12\frac{1}{4} + 4\frac{1}{5} \\
&\approx 0.4667.
\end{aligned}
$$

Quiz 3: R



3. We are given π_0. Find $\lim_{n\to\infty} \pi_n$.
   With probability $\alpha := 0.2\pi_0(1) + \pi_0(2) + \pi_0(3)$, the MC ends up in $\{2,3\}$.
   With probability $1-\alpha$, it ends up in state 4.
   If it is in $\{2,3\}$, the probability that it is in state 2 converges to

$$\frac{0.2}{0.2+0.6} = 0.25.$$

   Hence, the limiting distribution is

$$[0, 0.25\alpha, 0.75\alpha, 1-\alpha].$$

Quiz 3: R

4. A bag has n red and n blue balls. You pick two balls (no replacement). Let $X = 1$ if ball 1 is red and $X = -1$ otherwise. Define Y likewise for ball 2.
   $\rightarrow$ Are X and Y positively, negatively, or un- correlated?

   Clearly, negatively.

5. Calculate $cov(X, Y)$.

   $cov(X, Y) = E[XY] - E[X]E[Y]$

   $E[X] = E[Y]$, by symmetry

   $E[X] = 0$

   $E[XY] = Pr[X = Y] - Pr[X \neq Y] = 2Pr[X = Y] - 1$

   $Pr[X = Y] = (n-1)/(2n-1)$

   E.g., if $X = +1$ = red, then Y is red w.p. $(n-1)/(2n-1)$

   $E[XY] = 2(n-1)/(2n-1) - 1 = -1/(2n-1) = cov(X, Y)$.

6. What is $L[Y|X]$? $L[Y|X] = -\frac{1}{2n-1}X$. Indeed, $var(X) = 1$, obviously!

Quiz 3: R

7. A bag has n red and n blue balls. You pick two balls (no replacement). Let $X = 1$ if ball 1 is red and $X = -1$ otherwise. Define Y likewise for ball 2.
   Calculate $E[Y|X]$.

Since X takes only two values, any $g(X)$ is linear in X.
Hence, $E[Y|X] = L[Y|X]$.

Alternatively, Let $\alpha = Pr[X = Y] = (n-1)(2n-1)$.Then,

$$E[Y|X = 1] = \alpha - (1-\alpha) = 2\alpha - 1,$$
$$E[Y|X = -1] = -\alpha + (1-\alpha) = 1 - 2\alpha.$$

Thus,

$$E[Y|X] = (2\alpha - 1)X = -\frac{1}{2n-1}X.$$

Common Mistakes

- $\Omega = \{1,2,3\}$. Define X, Y with $cov(X,Y) = 0$ and X, Y not independent.

  Let $X = 0, Y = 1$. No: They are independent.

  Let $X(1) = -1, X(2) = 0, X(1) = 1, Y(1) = 0, Y(2) = 1, Y(3) = 0$.
- $3 \times 3.5 = 12.5$. No.
- $E[X^2] = E[X]^2$. No.
- $X = B(n,p) \implies var(X) = n^2 p(1-p)$. No.
- $E[X] = E[X|A] + E[X|\bar{A}]$. No.
- $\sum_{n=0}^{\infty} a^n = 1/a$. No.
- CS70 is difficult. No.
- I will do poorly on the final. No.
- Walrand is really weird. Probably!.

Thanks and Best Wishes!