Pre-Lecture

1. Homework party and office hour schedule is online, http://inst.eecs.berkeley.edu/~cs70/sp16/weekly.html.
   Check the time and location...will be updating.
   First homework party tonight: 6-9pm Cory 521!
2. Homework 1 is due Thursday 10pm (with an additional one-hour buffer period).
   Check Gradescope today to see if you have access to the course.
   And we get...
   "Base Case".
   If not, email name/SID/email to cs70@inst.eecs.berkeley.edu
   All students must do this homework, regardless of grading option choice.
3. Exam conflict? Please fill out the following the form on piazza at @105 by Feb 1, 2016.

Induction

The canonical way of proving statements of the form

\[(\forall k \in \mathbb{N})(P(k))\]

- For all natural numbers \(n, 1 + 2 + \ldots + n = \frac{n(n+1)}{2}\).
- For all \(n \in \mathbb{N}, n^2 - n\) is divisible by 3.
- The sum of the first \(n\) odd integers is a perfect square.

The basic form

- Prove \(P(0)\), "Base Case".
- \(P(k) \implies P(k+1)\)
  - Assume \(P(k)\), "Induction Hypothesis"
  - Prove \(P(k+1)\), "Induction Step."

\(P(n)\) true for all natural numbers \(n!!!\)

Get to use \(P(k)\) to prove \(P(k+1)!!!

Today.

Principle of Induction.

\[P(0) \land (\forall n \in \mathbb{N})P(n) \implies P(n+1)\]

And we get...

\[(\forall n \in \mathbb{N})P(n)\]

...Yes for 0, and we can conclude Yes for 1.

...and we can conclude Yes for 2......

Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate \(P(n)\) for \(n = k\), \(P(k)\) is \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate, \(P(n)\) true for \(n = k + 1\)?

\[\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}\]

How about \(k + 2\). Same argument starting at \(k + 1\) works!

Induction Step. \(P(k) \implies P(k+1)\).

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \(P(0)\) is \(\sum_{i=0}^{0} i = 1 = \frac{(0)(0+1)}{2}\) Base Case.

Statement is true for \(n = 1\) \(P(1)\) is true

plus inductive step \(\implies\) true for \(n = 2\) \((P(1) \implies P(2)) \implies P(2)\) \(\implies P(1)\)

...true for \(n = k \implies\) true for \(n = k + 1\) \((P(k) \implies P(k+1)) \implies P(k+1)\)

Predicate, \(P(n)\), True for all natural numbers! Proof by Induction.

Notes visualization

Note’s visualization: an infinite sequence of dominos.

Prove they all fall down:

- \(P(0) = \) “First domino falls”
- \((\forall k) P(k) \implies P(k+1):\) “\(k\)th domino falls implies that \(k + 1\)st domino falls”

Climb an infinite ladder?

\[P(0) \implies P(1) \implies P(2) \implies P(3) \ldots\]

\[\forall n \in \mathbb{N} P(n)\]

Your favorite example of forever... or the natural numbers...
Again: Simple induction proof.

Theorem: For all natural numbers \( n \), \( 0 + 1 + 2 \cdots + n = \frac{n(n+1)}{2} \)

Base Case: Does \( 0 = \frac{0(0+1)}{2} \) ? Yes.

Induction Step: Show \( \forall k \geq 0, P(k) \Rightarrow P(k+1) \)

Induction Hypothesis: \( P(k) = 1 + \cdots + k = \frac{k(k+1)}{2} \)

\[
1 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)
\]

\[
= \frac{k^2 + k + 2(k+1)}{2}
\]

\[
= \frac{k^2 + 3k + 2}{2}
\]

\[
= \frac{(k+1)(k+2)}{2}
\]

\( P(k+1) \). By principle of induction...

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Another Induction Proof.

Theorem: For every \( n \in N \), \( n^2 - n \) is divisible by 3. (\( 3|(n^2 - n) \).)

Proof: By induction.

Base Case: \( P(0) \) is \( (0^2) - 0 \) is divisible by 3. Yes!

Induction Step: \( \forall k \in N, P(k) \Rightarrow P(k+1) \)

Induction Hypothesis: \( k^2 - k \) is divisible by 3.

or \( k^3 - k = 3q \) for some integer \( q \).

\[
(k + 1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - (k+1)
\]

\[
= k^3 + 3k^2 + 2k
\]

\[
= (k^3 - k) + 3k^2 + 3k
\]

Subtract/add \( k \)

\[
= 3q + 3(k^2 + k)
\]

Induction Hyp. Factor.

\[
= 3(q + k^2 + k)
\]

(\( k + 1, k+2 \))

\[
= (k+1)^2 - (k+1) = 3(q + k^2 + k).
\]

Thus, \( \forall k \in N, P(k) \Rightarrow P(k+1) \)

Thus, theorem holds by induction.

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Four Color Theorem.

Theorem: Any map can be colored so that those regions that share an edge have different colors.

Check Out: “Four corners”.

States connected at a point, can have same color.
(Couldn’t find a map where they did though.)


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Strengthening Induction Hypothesis.

Theorem: The sum of the first \( n \) odd numbers is a perfect square.

\( \sum_{i=1}^{n} (2i-1) = n^2 \)

Base Case: \( n = 1 \)

\( 1^2 = 1 \)

Induction Hypothesis: Sum of first \( k \) odds is perfect square \( k^2 \).

Induction Step: 1. The \( (k+1) \text{st} \) odd number is \( 2k + 1 \).

2. Sum of the first \( k+1 \) odds is \( \sum_{i=1}^{k+1} (2i-1) = k^2 + 2k + 1 \)

3. \( k^2 + 2k + 1 = (k+1)^2 \)

Algorithm gives \( P(k) \Rightarrow P(k+1) \).
Hole can be anywhere!

**Theorem:** Can tile the $2^n \times 2^n$ square to leave a hole adjacent anywhere.

**Better theorem:** Better induction hypothesis!

**Base case:** Sure. A tile is fine.

**Induction Step:** Use induction hypothesis in each.

**Strong Induction:**

**Theorem:** Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

**Definition:** A prime $n$ has exactly 2 factors 1 and $n$.

**Base Case:** $n = 2$.

**Induction Step:**

Let $Q(k) = P(0) \land P(1) \land \cdots \land P(k)$.

By the induction principle:

- If $Q(0)$, and $(\forall k \in N)(Q(k) \implies Q(k+1))$
- Then $(\forall k \in N)(P(k))$

$Q(0)$ is true, and $P(k)$ is true for all $k$.

Strong Induction Principle:

If $Q(0)$ and $(\forall k \in N)Q(k) = P(k)$,

then $(\forall k \in N)Q(k) = P(k)$.

Let $Q(0) = P(0) \land P(1) \land \cdots \land P(k)$.

By the induction principle:

- If $Q(0)$, and $(\forall k \in N)(Q(k) \implies Q(k+1))$
- Then $(\forall k \in N)(P(k))$

$Q(0)$ is true, and $Q(k)$ is true for all $k$.

**Strong Induction Principle:**

If $P(0)$ and $(\forall k \in N)(P(0) \land \cdots \land P(k) \implies P(k+1))$

then $(\forall k \in N)(P(0) \land \cdots \land P(k)) \implies P(k+1)$.

Hole have to be there? Maybe just one?

**Theorem:** Any tiling of $2^n \times 2^n$ square has to have one hole.

**Proof:** The remainder of $2^{2n}$ divided by 3 is 1.

**Base case:** true for $k = 0$. $2^0 = 1$

**Ind Hyp:** $2^{2k} = 3a + 1$ for integer $a$.

$2^{2(k+1)} = 2^{2k} \times 2^2 = 4 \cdot 2^{2k} = 4 \cdot (3a + 1) = 12a + 3 + 1 = 3(4a + 1) + 1$

$a$ integer $\implies (4a + 1)$ is an integer.

Strong Induction.

**Theorem:** Every natural number $n > 1$ can be written as a product of primes.

**Definition:** A prime $n$ has exactly 2 factors 1 and $n$.

**Base Case:** $n = 2$.

**Induction Step:**

$P(n) = \text{"n can be written as a product of primes. "}$

Either $n + 1$ is a prime or $n + 1 = a \cdot b$ where $1 < a, b < n + 1$.

$P(n)$ says nothing about $a, b$!

**Strong Induction Principle:** If $P(0)$ and $(\forall k \in N)(P(0) \land \cdots \land P(k)) \implies P(k + 1))$, then $(\forall k \in N)(P(k))$.

Strong induction hypothesis: "a and b are products of primes"

$n + 1 = a \cdot b = (\text{factorization of a})(\text{factorization of b})$,

$n + 1$ can be written as the product of the prime factors!
Well Ordering Principle and Induction.

If (∀n)P(n) is not true, then (∃n)¬P(n).

Consider smallest m, with ¬P(m), m ≥ 0

P(m−1) → P(m) must be false (assuming P(0) holds.)

This is a proof of the induction principle!

I.e.,

(¬∀n)P(n) =→ (∃n)(¬P(n−1) =→ P(n)).

(Contrapositive of Induction principle (assuming P(0))

It assumes that there is a smallest m where P(m) does not hold.

The Well ordering principle states that for any subset of the natural numbers there is a smallest element.

Smallest may not be what you expect: the well ordering principal holds for rationals but with different ordering!!

E.g. Reduced form is “smallest” representation of a rational number a/b.

Tournaments have short cycles

Def: A round robin tournament on n players: every player p plays every other player q, and either p → q (p beats q) or q → p (q beats p.)

Def: A cycle: a sequence of p₁,..., pₖ, p₁ → p₂,... pₖ and pₖ → p₁.

Theorem: Any tournament that has a cycle has a cycle of length 3.

Tournaments have long paths.

Def: A Hamiltonian path: a sequence p₁,..., pₙ, (1 ≤ i < n) pᵢ → pᵢ₊₁.

Base: True for two vertices.

(Also for one, but two is more useful as base case!)

Tournament on n + 1 people.

Remove arbitrary person → yield tournament on n − 1 people.

(Result specified for each remaining pair from original tournament.)

By induction hypothesis: There is a sequence p₁,..., pₙ contains all the people

where pᵢ → pᵢ₊₁

If p is big winner, put at beginning.

If not, find first place i, where p beats pᵢ.

p₁,..., pᵢ−₁, p, pᵢ,..., pₙ is hamiltonion path.

If no place, place at the end.

Horses of the same color...

Theorem: All horses have the same color.

Base Case: P(1) - trivially true.

New Base Case: P(2): there are two horses with same color.

Induction Hypothesis: P(k) - Any k horses have the same color.

Induction step P(k + 1)?

First k have same color by P(k).

Second k have same color by P(k).

A horse in the middle in common! 1, 2, 3,..., k + 1

All k must have different color!!

How about P(1) =⇒ P(2)?

Fix base case.

...Still doesn’t work!!

(There are two horses is ≠ For all two horses!!!)

Of course it doesn’t work.

As we will see, it is more subtle to catch errors in proofs of correct theorems!!

Thm: All natural numbers are interesting.

0 is interesting...

Let n be the first uninteresting number.

But n - 1 is interesting and n is uninteresting, so this is the first uninteresting number.

But this is interesting.

Thus, there is no smallest uninteresting natural number.

Thus: All natural numbers are interesting.
Strong Induction and Recursion.

Thm: For every natural number \( n \geq 12, n = 4x + 5y. \)

Instead of proof, let's write some code!

```python
def find_x_y(n):
def find_x_y(n):
    if (n==12): return (3,0)
    elif (n==13): return(2,1)
    elif (n==14): return(1,2)
    else:
        (x',y') = find_x_y(n-4)
        return(x'+1,y')
```

Base cases: \( P(12), P(13), P(14), P(15). \) Yes.

Strong Induction step:

Recursive call is correct: \( P(n-4) \Longrightarrow P(n). \)

\( n-4 = 4x'+5y' \Longrightarrow n = 4(x'+1) + 5y' \)

Slight differences: showed for all \( n \geq 16 \) that \( \sum_{i=4}^{n-4} P(i) \Longrightarrow P(n). \)

Summary: principle of induction.

Today: More induction.

\( (P(0) \land ((\forall k \in \mathbb{N})(P(k) \Longrightarrow P(k+1)))) \Longrightarrow (\forall n \in \mathbb{N})(P(n)) \)

Statement to prove: \( P(n) \) for \( n \) starting from \( n_0 \)

Base Case: Prove \( P(n_0). \)

Statement is proven!

Strong Induction:

\( (P(1) \land ((\forall n \in \mathbb{N})((n \geq 1) \land P(n)) \Longrightarrow P(n+1)))) \)

Also Today: strengthened induction hypothesis.

**Strengthen theorem statement.**

Sum of first \( n \) odds is \( n^2. \)

Hole anywhere.

Not same as strong induction.

Induction = Recursion.

Summary: principle of induction.

\( (\forall n \geq 12, n = 4x + 5y). \)

Instead of proof, let's write some code!

```python
def find_x_y(n):
def find_x_y(n):
    if (n==12): return (3,0)
    elif (n==13): return(2,1)
    elif (n==14): return(1,2)
    else:
        (x',y') = find_x_y(n-4)
        return(x'+1,y')
```

Base cases: \( P(12), P(13), P(14), P(15). \) Yes.

Strong Induction step:

Recursive call is correct: \( P(n-4) \Longrightarrow P(n). \)

\( n-4 = 4x'+5y' \Longrightarrow n = 4(x'+1) + 5y' \)

Slight differences: showed for all \( n \geq 16 \) that \( \sum_{i=4}^{n-4} P(i) \Longrightarrow P(n). \)

Summary: principle of induction.

\( (P(0) \land ((\forall k \in \mathbb{N})(P(k) \Longrightarrow P(k+1)))) \Longrightarrow (\forall n \in \mathbb{N})(P(n)) \)

Variations:

\( (P(1) \land ((\forall n \in \mathbb{N})((n \geq 1) \land P(n)) \Longrightarrow P(n+1)))) \)

Statement to prove: \( P(n) \) for \( n \) starting from \( n_0 \)

Base Case: Prove \( P(n_0). \)

Statement is proven!