Pre-Lecture

1. Homework party and office hour schedule is online.
   http://inst.eecs.berkeley.edu/~cs70/sp16/weekly.html.
   Check the time and location..will be updating.
   First homework party tonight: 6-9pm Cory 521!
2. Homework 1 is due Thursday 10pm (with an additional one-hour buffer period).
   Check Gradescope today to see if you have access to the course.
   If not, email name/SID/email to cs70@inst.eecs.berkeley.edu
   All students must do this homework, regardless of grading option choice.
3. Exam conflict? Please fill out the following the form on piazza at @105 by Feb 1, 2016.

Today.

Principle of Induction.

$P(0) \wedge (\forall n \in \mathbb{N}) P(n) \implies P(n+1)$

And we get...

$(\forall n \in \mathbb{N}) P(n)$.

...Yes for 0, and we can conclude Yes for 1...
and we can conclude Yes for 2.......

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$.

How about $k+2$. Same argument starting at $k+1$ works!
Induction Step. $P(k) \implies P(k+1)$.

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = 1 = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true
plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$
plus inductive step $\implies$ true for $n = 2$ $(P(1) \wedge (P(1) \implies P(2))) \implies P(2)$
...
true for $n = k$ $\implies$ true for $n = k+1$ $(P(k) \wedge (P(k) \implies P(k+1))) \implies P(k+1)$
...

Predicate, $P(n)$, True for all natural numbers! **Proof by Induction.**

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first n odd integers is a perfect square.

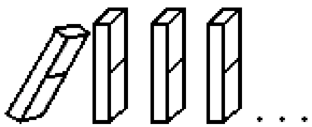
The basic form

- Prove $P(0)$. "Base Case".
- $P(k) \implies P(k+1)$
  - Assume $P(k)$, "Induction Hypothesis"
  - Prove $P(k+1)$. "Induction Step."

$P(n)$ true for all natural numbers n!!!
Get to use $P(k)$ to prove $P(k+1)$! ! ! !

Notes visualization

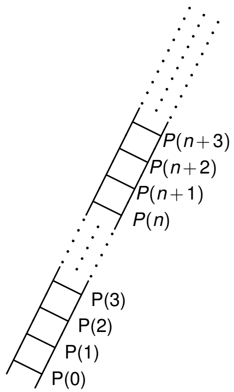
Note's visualization: an infinite sequence of dominos.



Prove they all fall down;

- $P(0)$ = "First domino falls"
- $(\forall k)\ P(k) \implies P(k+1)$:
  "kth domino falls implies that $k+1$st domino falls"

Climb an infinite ladder?



$P(0)$
$\forall k, P(k) \implies P(k+1)$
$P(0) \implies P(1) \implies P(2) \implies P(3) \ldots$
$(\forall n \in N) P(n)$

Your favorite example of forever..or the natural numbers...

Again: Simple induction proof.

Theorem: For all natural numbers n, $0+1+2\cdots n = \frac{n(n+1)}{2}$

Base Case: Does $0 = \frac{0(0+1)}{2}$? Yes.

Induction Step: Show $\forall k \geq 0, P(k) \implies P(k+1)$
Induction Hypothesis: $P(k) = 1+\cdots+k = \frac{k(k+1)}{2}$

$$\begin{aligned} 1+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1) \\ &= \frac{k^2+k+2(k+1)}{2} \\ &= \frac{k^2+3k+2}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

$P(k+1)$!. By principle of induction... □

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. ($3|(n^3-n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3)-0$" is divisible by 3. Yes!
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$
Induction Hypothesis: $k^3 - k$ is divisible by 3.
or $k^3 - k = 3q$ for some integer q.

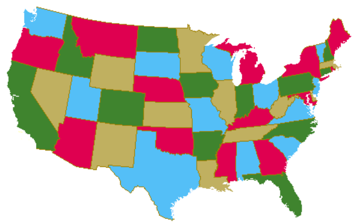
$$\begin{aligned} (k+1)^3-(k+1) &= k^3+3k^2+3k+1-(k+1) \\ &= k^3+3k^2+2k \\ &= (k^3-k)+3k^2+3k && \text{Subtract/add } k \\ &= 3q+3(k^2+k) && \text{Induction Hyp. Factor.} \\ &= 3(q+k^2+k) && \text{(Un)Distributive + over } \times \end{aligned}$$

Or $(k+1)^3-(k+1) = 3(q+k^2+k)$.

$(q+k^2+k)$ is integer (closed under addition and multiplication).
$\implies (k+1)^3-(k+1)$ is divisible by 3.
Thus, $(\forall k \in N) P(k) \implies P(k+1)$
Thus, theorem holds by induction. □

Four Color Theorem.

Theorem: Any map can be colored so that those regions that share an edge have different colors.

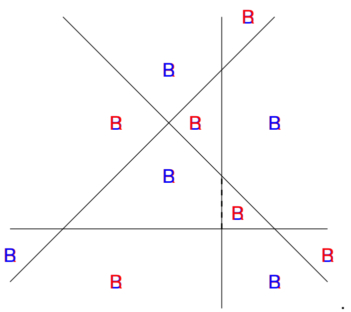


Check Out: "Four corners".
States connected at a point, can have same color.
(Couldn't find a map where they did though.)

Quick Test: Which states? Utah. Colorado. New Mexico. Arizona.

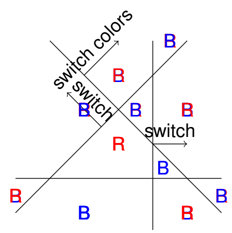
Two color theorem: example.

Any map formed by dividing the planeM into regions by drawing straight lines can be properly colored with two colors.



Fact: Swapping red and blue gives another valid colors.

Two color theorem: proof illustration.



Base Case.
1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.
(Fixes conflicts along line, and makes no new ones.)

Algorithm gives $P(k) \implies P(k+1)$. □

Strenthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.
Theorem: The sum of the first n odd numbers is n^2.

kth odd number is $2(k-1)+1$.

Base Case: 1 (1th odd number) is 1^2.

Induction Hypothesis: Sum of first k odds is perfect square $a^2 = k^2$.

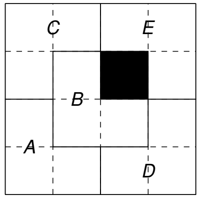
Induction Step:
1. The $(k+1)$st odd number is $2k+1$.
2. Sum of the first $k+1$ odds is $a^2+2k+1 = k^2+2k+1$
3. $k^2+2k+1 = (k+1)^2$
... P(k+1)!

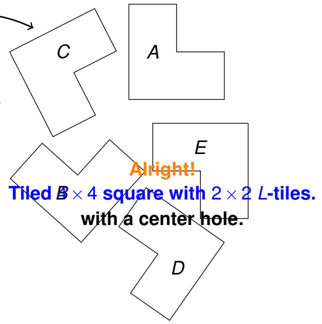
□

Tiling Cory Hall Courtyard.

Use these L-tiles.

To Tile this 4×4 courtyard.





Can we tile any $2^n \times 2^n$ with L-tiles (with a hole) for every n!

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$

Ind Hyp: $2^{2k} = 3a + 1$ for integer a.

$$\begin{aligned} 2^{2(k+1)} &= 2^{2k} * 2^2 \\ &= 4 * 2^{2k} \\ &= 4 * (3a+1) \\ &= 12a + 3 + 1 \\ &= 3(4a+1) + 1 \end{aligned}$$

a integer $\implies$ $(4a+1)$ is an integer. $\square$

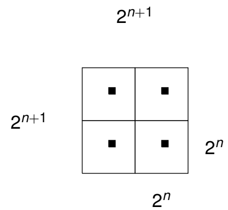
Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

Proof:

Base case: A single tile works fine.
The hole is adjacent to the center of the 2×2 square.

Induction Hypothesis:
Any $2^n \times 2^n$ square can be tiled with a hole at the center.



What to do now???

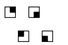
Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere*.

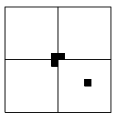
Better theorem ...better induction hypothesis!

Base case: Sure. A tile is fine.

Flipping the orientation can leave hole anywhere.



Induction Hypothesis:
"Any $2^n \times 2^n$ square can be tiled with a hole **anywhere.**"
Consider $2^{n+1} \times 2^{n+1}$ square.



Use induction hypothesis in each.

Use L-tile and ... we are done. $\square$

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.
Definition: A prime n has exactly 2 factors 1 and n.
Base Case: $n = 2$.
Induction Step:
$P(n) =$ "n can be written as a product of primes. "
Either $n+1$ is a prime or $n+1 = a \cdot b$ where $1 < a, b < n+1$.
$P(n)$ says nothing about a, b!

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

$$P(0) \implies P(1) \implies P(2) \implies P(3) \implies \cdots$$

Strong induction hypothesis: "a and b are products of primes"

$\implies$ "$n+1 = a \cdot b =$ (factorization of a)(factorization of b)"
$n+1$ can be written as the product of the prime factors! $\square$

Induction $\implies$ Strong Induction.

Let $Q(k) = P(0) \wedge P(1) \cdots P(k)$.

By the induction principle:
"If $Q(0)$, and $(\forall k \in N)(Q(k) \implies Q(k+1))$ then $(\forall k \in N)(Q(k))$"

Also, $Q(0) \equiv P(0)$, and $(\forall k \in N)(Q(k)) \equiv (\forall k \in N)(P(k))$

$(\forall k \in N)(Q(k) \implies Q(k+1))$
$\equiv (\forall k \in N)((P(0) \cdots \wedge P(k)) \implies (P(0) \cdots P(k) \wedge P(k+1)))$
$\equiv (\forall k \in N)((P(0) \cdots \wedge P(k)) \implies P(k+1))$

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

Consider smallest m, with $\neg P(m)$, $m \geq 0$

$P(m-1) \implies P(m)$ must be false (assuming $P(0)$ holds.)

This is a proof of the induction principle!
I.e.,

$$(\neg\forall n)P(n) \implies ((\exists n)\neg(P(n-1) \implies P(n)).$$

(Contrapositive of Induction principle (assuming $P(0)$)

It assumes that there is a smallest m where $P(m)$ does not hold.

The **Well ordering principle** states that for any subset of the natural numbers there is a smallest element.

Smallest may not be what you expect: the well ordering principal holds for rationals but with different ordering!!

E.g. Reduced form is "smallest" representation of a rational number a/b.

Thm: All natural numbers are interesting.

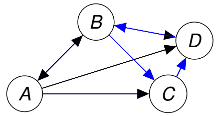
0 is interesting...
Let n be the first uninteresting number.
But $n-1$ is interesting and n is uninteresting,
so this is the first uninteresting number.
But this is interesting.
Thus, there is no smallest uninteresting natural number.

Thus: All natural numbers are interesting.

Tournaments have short cycles

Def: A **round robin tournament on** n **players**: every player p plays every other player q, and either $p \to q$ (p beats q) or $q \to p$ (q beats p.)

Def: A **cycle**: a sequence of $p_1, \ldots, p_k$, $p_i \to p_{i+1}$ and $p_k \to p_1$.



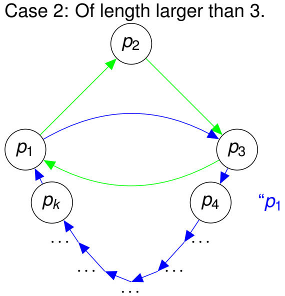
Theorem: Any tournament that has a cycle has a cycle of length 3.

Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length k.

Case 1: Of length 3. Done.

Case 2: Of length larger than 3.



"$p_3 \to p_1$" $\implies$ 3 cycle

Contradiction.

"$p_1 \to p_3$" $\implies$ $k-1$ length cycle!

Contradiction!

Tournaments have long paths.

Def: A **round robin tournament on** n **players**: every player p plays every other player q, and either $p \to q$ (p beats q) or $q \to q$ (q beats q.)

Def: A **Hamiltonian path**: a sequence
$p_1, \ldots, p_n$, $(\forall i, 0 \leq i < n)\ p_i \to p_{i+1}$.

Base: True for two vertices.
(Also for one, but two is more useful as base case!)

Tournament on $n+1$ people,
Remove arbitrary person $\to$ yield tournament on $n-1$ people.
(Result specified for each remaining pair from original tournament.)

By induction hypothesis: There is a sequence $p_1, \ldots, p_n$
contains all the people
where $p_i \to p_{i+1}$

If p is big winner, put at beginning.
If not, find first place i, where p beats p_i.
$p_1, \ldots, p_{i-1}, p, p_i, \ldots p_n$ is hamiltonion path.
If no place, place at the end.

$\square$

Horses of the same color...

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.
New Base Case: $P(2)$: there are two horses with same color.

Induction Hypothesis: $P(k)$ - Any k horses have the same color.

Induction step $P(k+1)$?
First k have same color by $P(k)$. $1, 2, 3, \ldots, k, k+1$
Second k have same color by $P(k)$. $1, 2, 3, \ldots, k, k+1$
A horse in the middle in common! $1, 2, 3, \ldots, k, k+1$
All k must have the same color. No horse in common! $1, 2, 3, \ldots, k, k+1$
How about $P(1) \implies P(2)$?
Fix base case.
...Still doesn't work!!
(There are two horses is $\not\equiv$ For all two horses!!!)

Of course it doesn't work.

As we will see, it is more subtle to catch errors in proofs of correct theorems!!

Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let's write some code!

```
def find-x-y(n):
   if (n==12) return (3,0)
   elif (n==13): return(2,1)
   elif (n==14): return(1,2)
   elif (n==15): return(0,3)
   else:
      (x',y') = find-x-y(n-4)
      return(x'+1,y')
```

Base cases: P(12) , P(13) , P(14) , P(15). Yes.

Strong Induction step:
Recursive call is correct: $P(n-4) \implies P(n)$.
$n-4 = 4x' + 5y' \implies n = 4(x'+1) + 5(y')$

Slight differences: showed for all $n \geq 16$ that $\wedge_{i=4}^{n-1} P(i) \implies P(n)$.

Summary: principle of induction.

Today: More induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.
Statement is proven!

Strong Induction:
$(P(0) \wedge ((\forall n \in N)(P(n)) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$

Also Today: strengthened induction hypothesis.

Strengthen theorem statement.
Sum of first n odds is n^2.
Hole anywhere.
Not same as strong induction.

Induction $\equiv$ Recursion.

Summary: principle of induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Variations:
$(P(0) \wedge ((\forall n \in N)(P(n) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$

$(P(1) \wedge ((\forall n \in N)((n \geq 1) \wedge P(n)) \implies P(n+1))))$
$\implies (\forall n \in N)((n \geq 1) \implies P(n))$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.
Statement is proven!