Pre-Lecture

1. Homework party and office hour schedule is online.
   http://inst.eecs.berkeley.edu/˜cs70/sp16/weekly.html.

   Check the time and location..will be updating.

   First homework party tonight: 6-9pm Cory 521!

2. Homework 1 is due Thursday 10pm (with an additional one-hour buffer period).
   Check Gradescope today to see if you have access to the course.

   If not, email name/SID/email to cs70@inst.eecs.berkeley.edu
   All students must do this homework, regardless of grading option choice.

3. Exam conflict? Please fill out the following the form on piazza at @105 by Feb 1, 2016.

Today.

Principle of Induction.

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$$P(0) \wedge (\forall n \in \mathbb{N}) P(n) \implies P(n+1)$$

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And we get...

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$$(\forall n \in \mathbb{N}) P(n).$$

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...Yes for 0,

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...Yes for 0, and we can conclude Yes for 1...
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...Yes for 0, and we can conclude Yes for 1...
and we can conclude Yes for 2...

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And we get...

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Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$

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Is predicate, $P(n)$ true for $n = k + 1$?

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Statement is true for $n = 0$

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plus inductive step

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plus inductive step $\implies$ true for $n = 1$

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plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$

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true for $n = k \implies$ true for $n = k+1$

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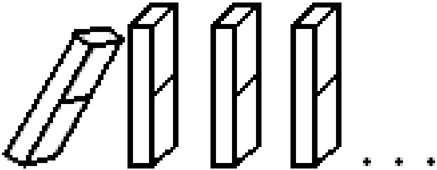
Notes visualization

Note’s visualization: an infinite sequence of dominos.



Prove they all fall down;

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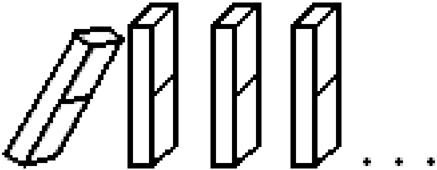

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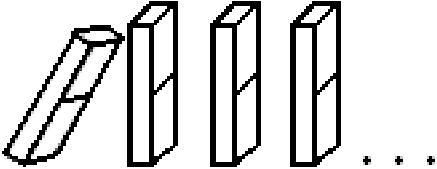

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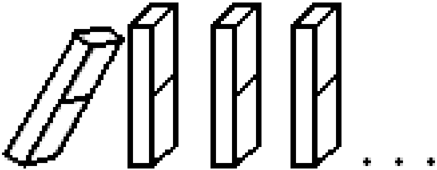

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Climb an infinite ladder?

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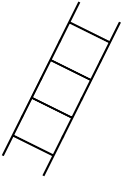

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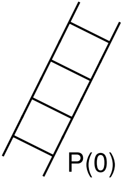

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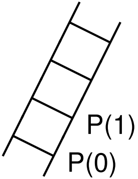

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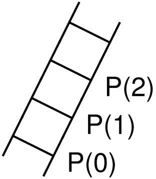

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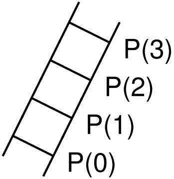

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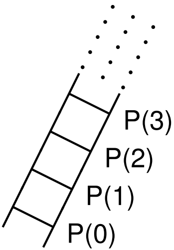

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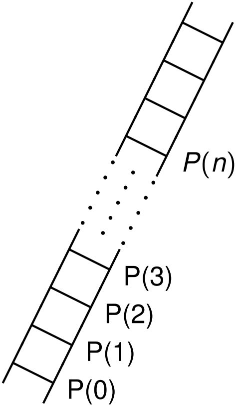

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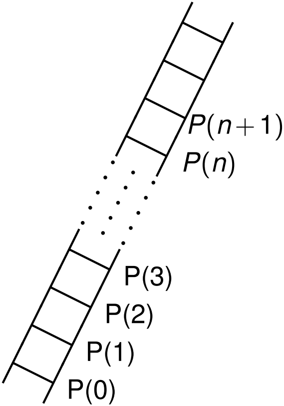

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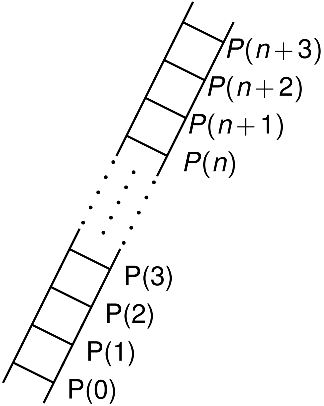

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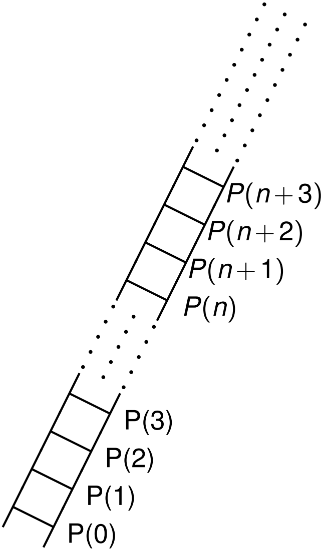

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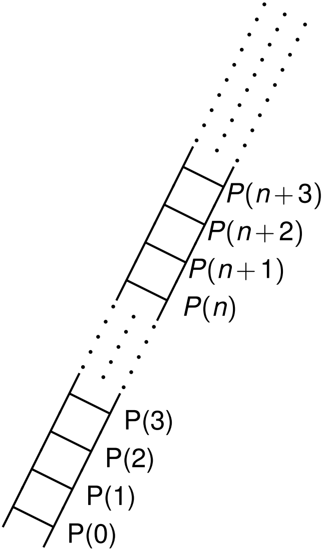

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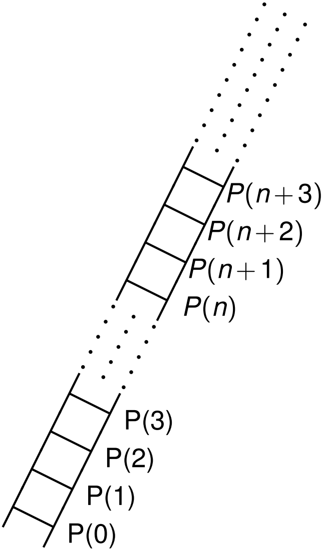

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1+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1) \\
&= \frac{k^2+k+2(k+1)}{2} \\
&= \frac{k^2+3k+2}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}
$$

$P(k+1)$!. By principle of induction...

Again: Simple induction proof.

Theorem: For all natural numbers n, $0+1+2\cdots n = \frac{n(n+1)}{2}$

Base Case: Does 0 = $\frac{0(0+1)}{2}$? Yes.

Induction Step: Show $\forall k \geq 0, P(k) \implies P(k+1)$
Induction Hypothesis: $P(k) = 1+\cdots+k = \frac{k(k+1)}{2}$

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$(k+1)^3 - (k+1)$

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$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - (k+1)$

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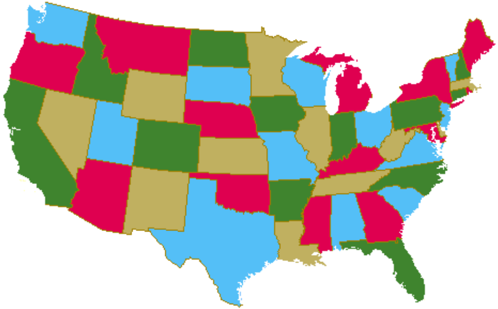
Four Color Theorem.

Theorem: Any map can be colored so that those regions that share an edge have different colors.



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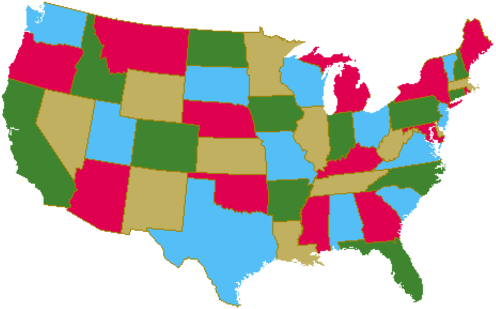

Check Out: “Four corners”.

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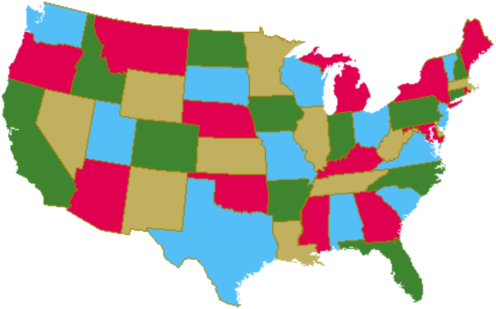

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States connected at a point, can have same color.

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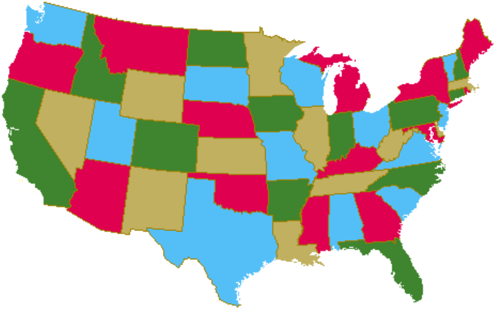

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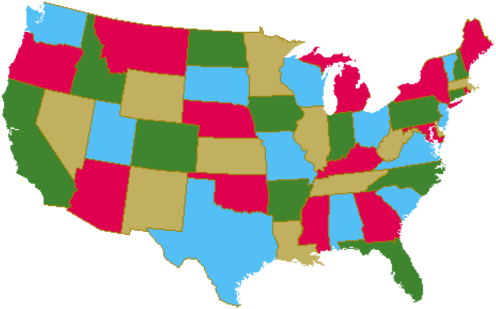

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Quick Test: Which states?

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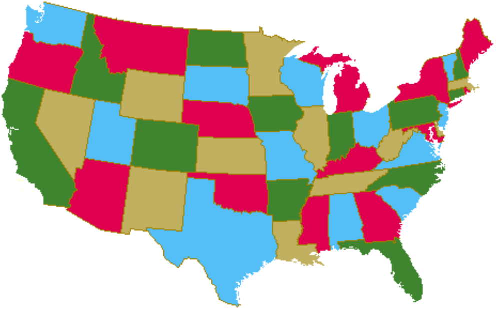

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Quick Test: Which states? Utah.

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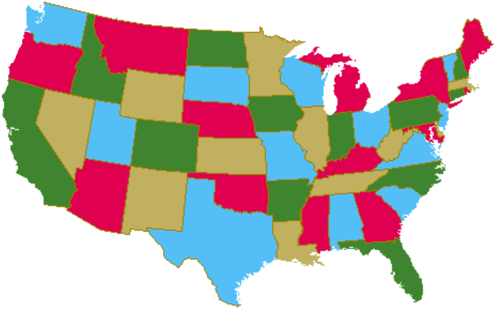

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Quick Test: Which states? Utah. Colorado.

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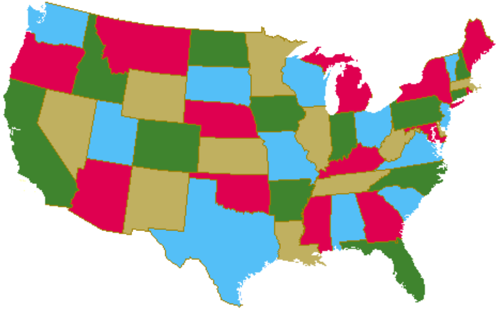

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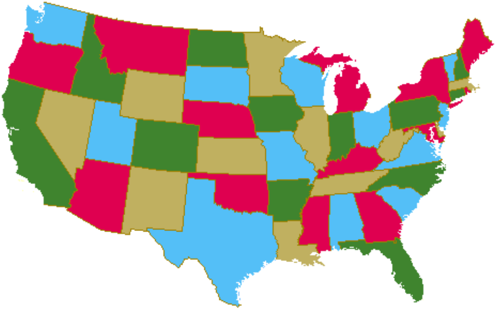

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Two color theorem: example.

Any map formed by dividing the planeM into regions by drawing straight lines can be properly colored with two colors.

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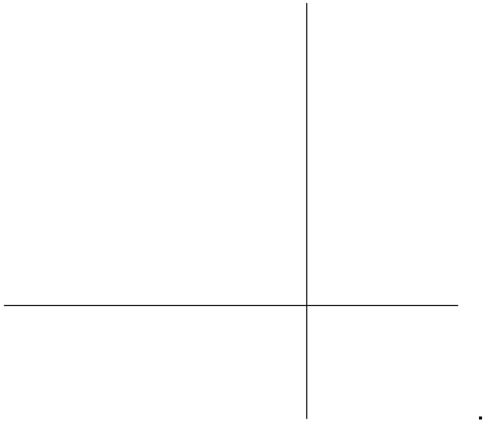

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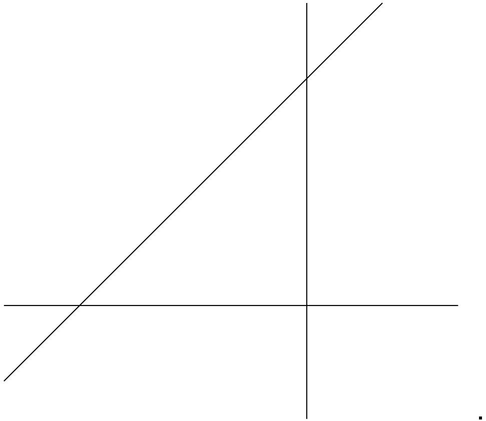

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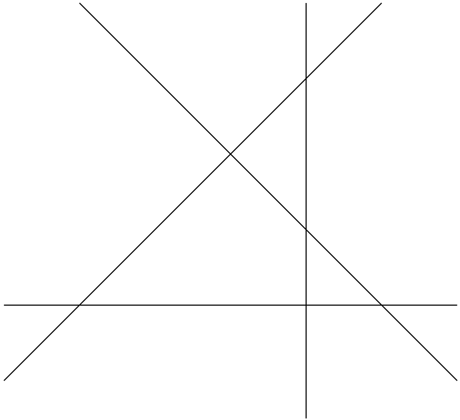

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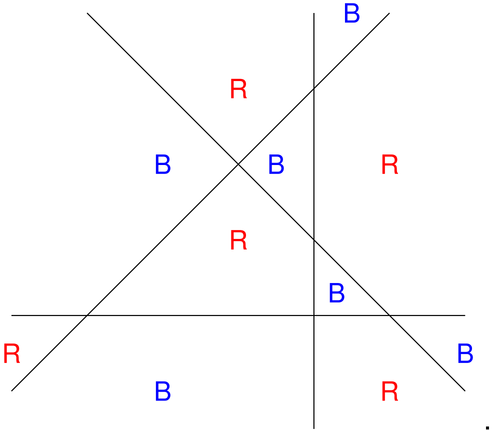

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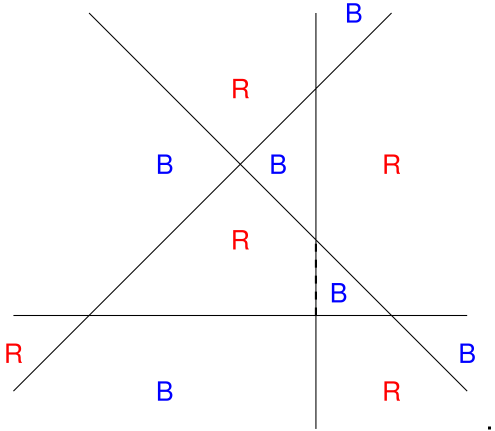

Fact: Swapping red and blue gives another valid colors.

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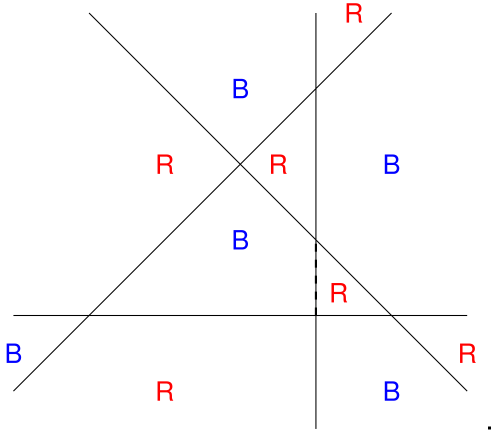
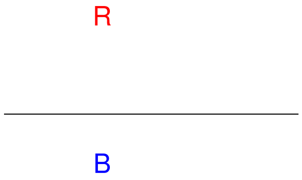


Fact: Swapping red and blue gives another valid colors.

Two color theorem: proof illustration.

Base Case.

Two color theorem: proof illustration.



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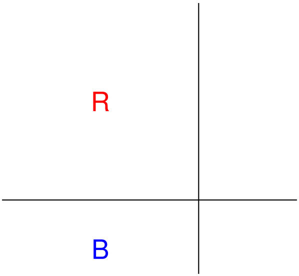
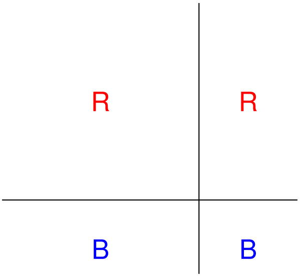


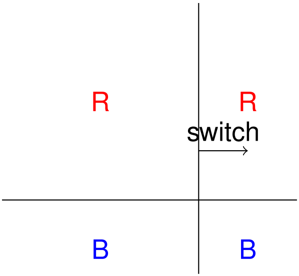
1. Add line.

Two color theorem: proof illustration.



1. Add line.
2. Get inherited color for split regions

Two color theorem: proof illustration.



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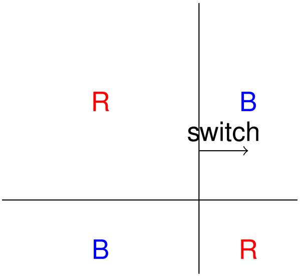
(Fixes conflicts along line, and makes no new ones.)

Two color theorem: proof illustration.



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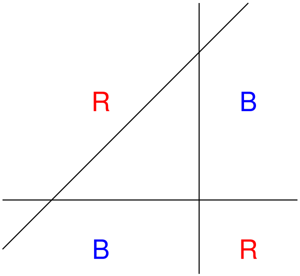

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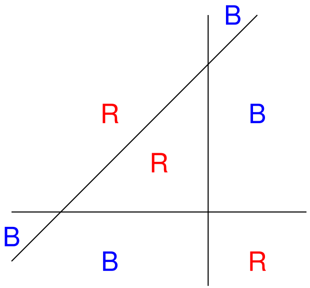

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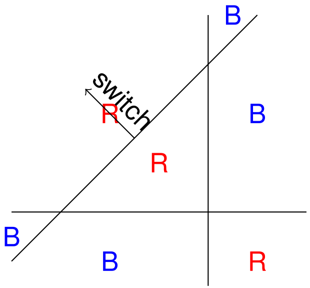

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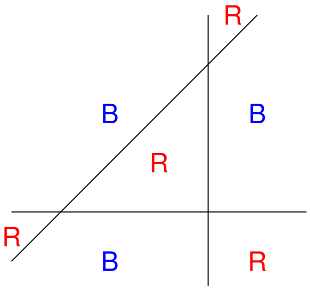

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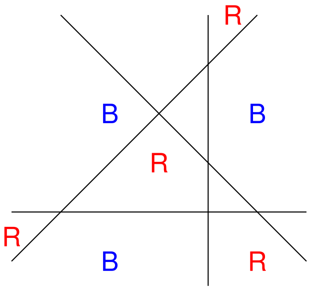

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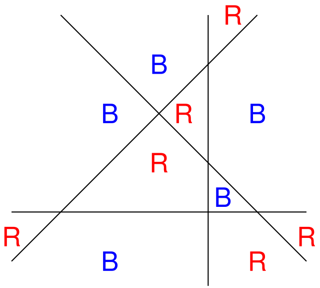

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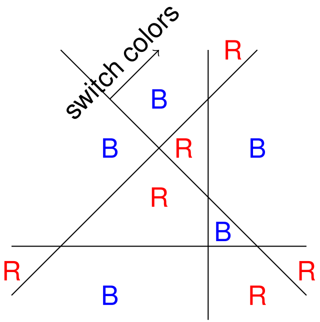

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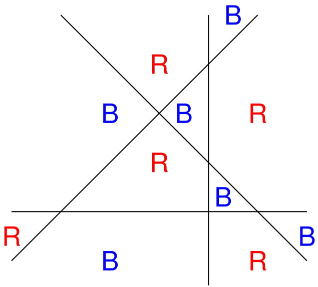

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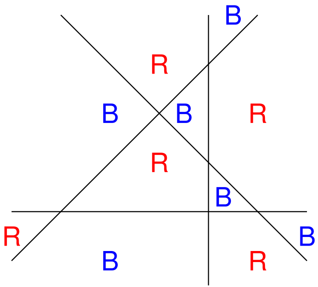

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Algorithm gives $P(k) \implies P(k+1)$.

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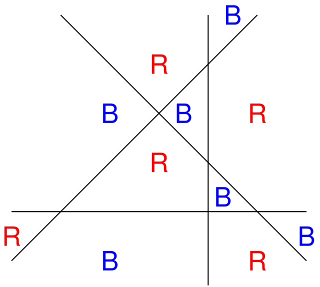

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Algorithm gives $P(k) \implies P(k+1)$. $\square$

Strenthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.

kth odd number is $2(k-1)+1$.

Base Case: 1 (1th odd number) is 1^2.

Induction Hypothesis: Sum of first k odds is perfect square a^2

Induction Step:

1. The $(k+1)$st odd number is $2k+1$.
2. Sum of the first $k+1$ odds is $a^2+2k+1 = k^2+2k+1$

□

Strenthening Induction Hypothesis.

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Theorem: The sum of the first n odd numbers is n^2.

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Induction Hypothesis: Sum of first k odds is perfect square $a^2 = k^2$.

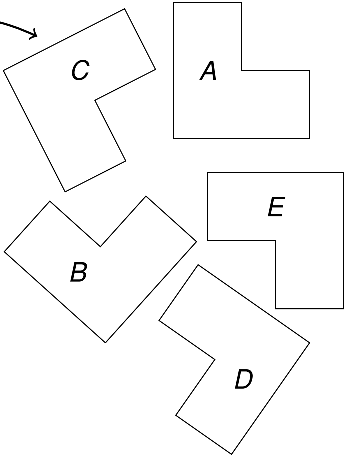
Induction Step:

1. The $(k+1)$st odd number is $2k+1$.
2. Sum of the first $k+1$ odds is $a^2+2k+1 = k^2+2k+1$
3. $k^2+2k+1 = (k+1)^2$
   ... P(k+1)!

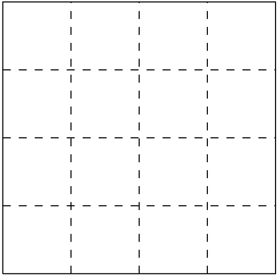
□

Tiling Cory Hall Courtyard.

Use these L-tiles.



To Tile this 4×4 courtyard.



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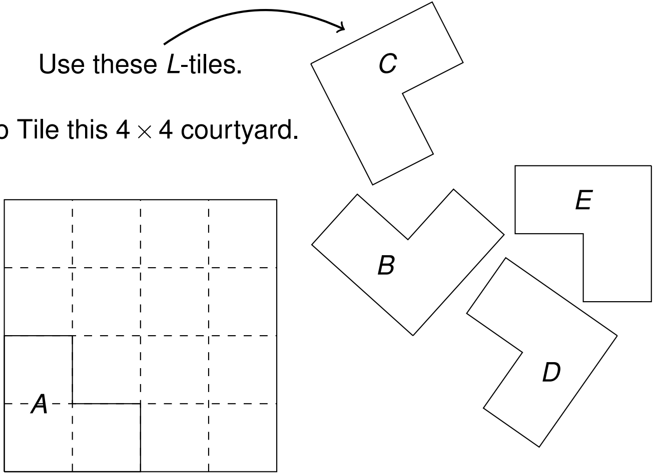

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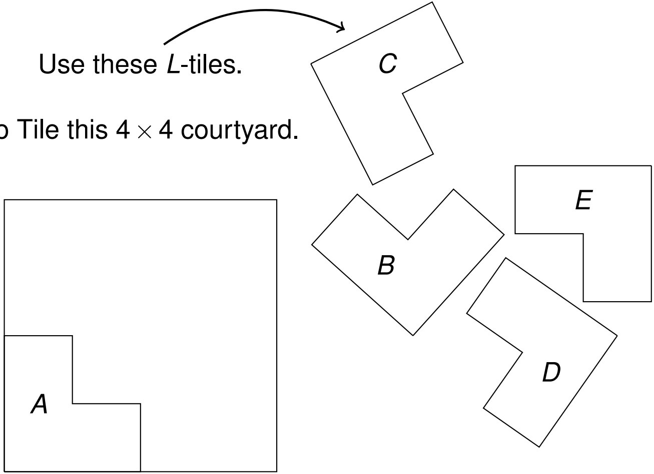

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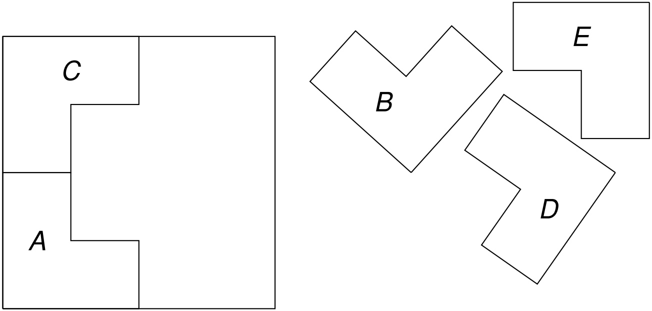

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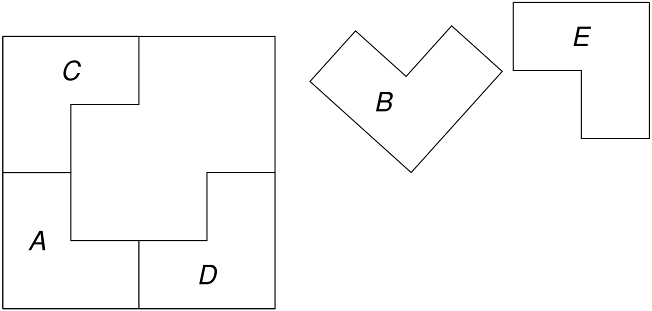

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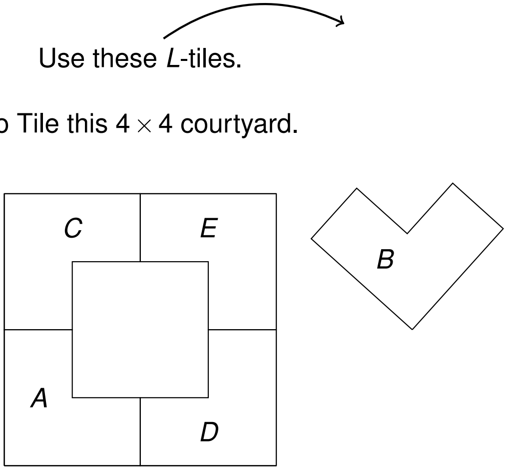

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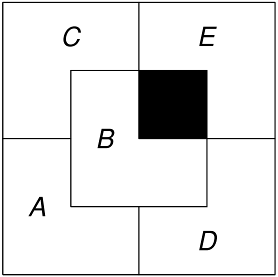

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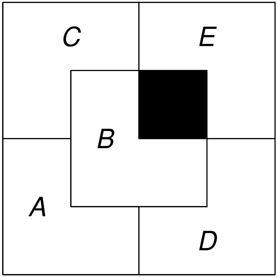

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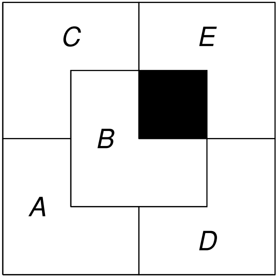


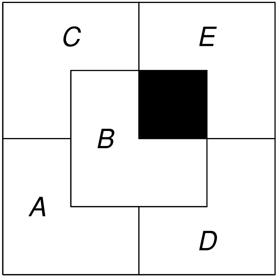
Alright!

Tiled 4×4 square with 2×2 L-tiles.

Tiling Cory Hall Courtyard.

Use these L-tiles.

To Tile this 4×4 courtyard.



Alright!

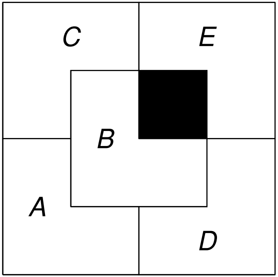
Tiled 4×4 square with 2×2 L-tiles.

with a center hole.

Tiling Cory Hall Courtyard.

Use these L-tiles.

To Tile this 4×4 courtyard.



Alright!

Tiled 4×4 square with 2×2 L-tiles.

with a center hole.

Can we tile any $2^n \times 2^n$ with L-tiles (with a hole)

Tiling Cory Hall Courtyard.

Use these L-tiles.

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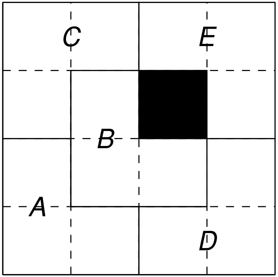

Alright!

Tiled 4×4 square with 2×2 L-tiles.

with a center hole.

Can we tile any $2^n \times 2^n$ with L-tiles (with a hole) for every n!

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

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Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$.

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Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$

Ind Hyp: $2^{2k} = 3a + 1$ for integer a.

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$$2^{2(k+1)}$$

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$$2^{2(k+1)} = 2^{2k} * 2^2$$

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$$\begin{aligned} 2^{2(k+1)} &= 2^{2k} * 2^2 \\ &= 4 * 2^{2k} \end{aligned}$$

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$$\begin{aligned} 2^{2(k+1)} &= 2^{2k} * 2^2 \\ &= 4 * 2^{2k} \\ &= 4 * (3a + 1) \\ &= 12a + 3 + 1 \end{aligned}$$

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$$
\begin{aligned}
2^{2(k+1)} &= 2^{2k} * 2^2 \\
&= 4 * 2^{2k} \\
&= 4 * (3a + 1) \\
&= 12a + 3 + 1 \\
&= 3(4a + 1) + 1
\end{aligned}
$$

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a integer

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a integer $\Longrightarrow$ $(4a + 1)$ is an integer.

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Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.
Proof:

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Base case: A single tile works fine.
The hole is adjacent to the center of the 2×2 square.

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Induction Hypothesis:
Any $2^n \times 2^n$ square can be tiled with a hole at the center.

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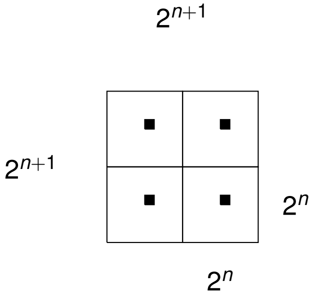

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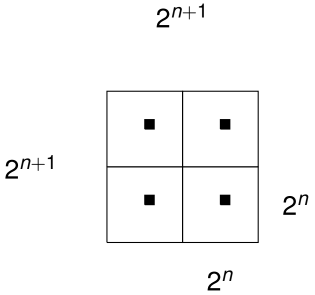

What to do now???

Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

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Better theorem

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Better theorem ...better induction hypothesis!

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Base case: Sure. A tile is fine.

Hole can be anywhere!

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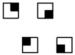
Base case: Sure. A tile is fine.

Flipping the orientation can leave hole anywhere.



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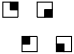

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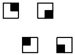
Flipping the orientation can leave hole anywhere. 

Induction Hypothesis:
"Any $2^n \times 2^n$ square can be tiled with a hole **anywhere.**"
Consider $2^{n+1} \times 2^{n+1}$ square.

Hole can be anywhere!

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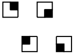

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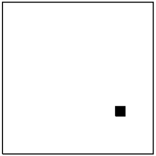

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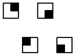

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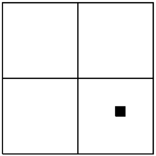

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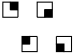

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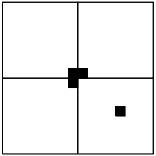

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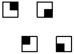

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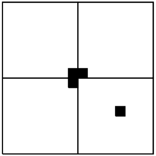

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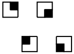

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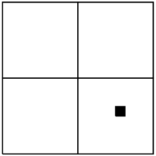

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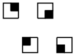

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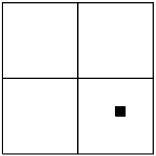

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E.g. Reduced form is “smallest” representation of a rational number a/b.

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Def: A **cycle**: a sequence of $p_1, \ldots, p_k$, $p_i \to p_{i+1}$ and $p_k \to p_1$.

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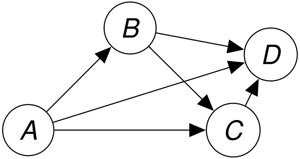

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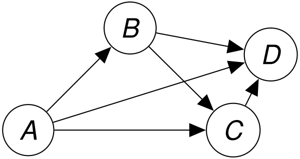

Theorem: Any tournament that has a cycle has a cycle of length 3.

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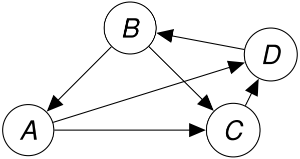

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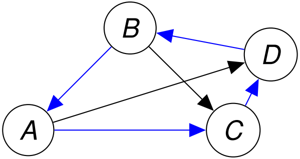

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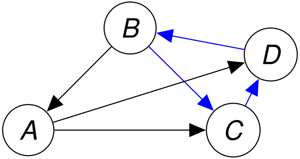

Theorem: Any tournament that has a cycle has a cycle of length 3.

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Assume the the **smallest cycle** is of length k.

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Case 1: Of length 3.

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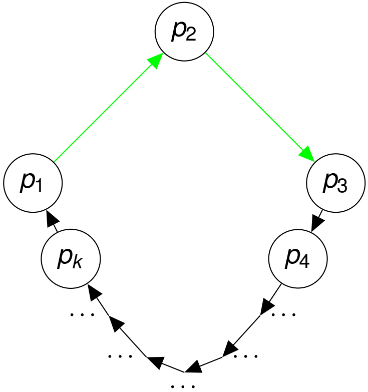
Case 1: Of length 3. Done.

Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length k.

Case 1: Of length 3. Done.

Case 2: Of length larger than 3.



"$p_3 \rightarrow p_1$" $\implies$ 3 cycle

Contradiction.

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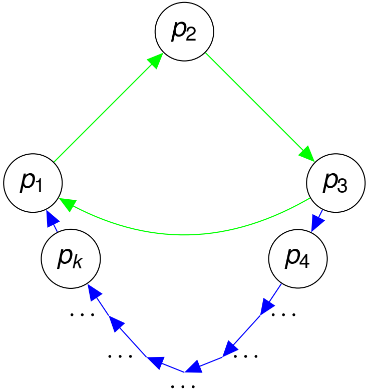

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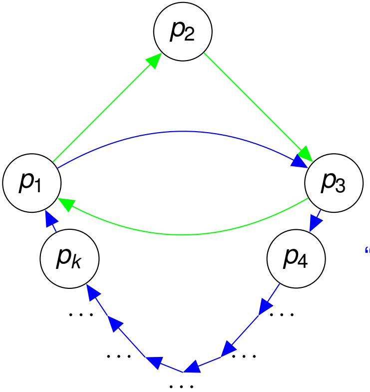

"$p_3 \to p_1$" $\implies$ 3 cycle

Contradiction.

"$p_1 \to p_3$" $\implies$ $k-1$ length cycle!

Contradiction!

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Tournament on $n+1$ people,
Remove arbitrary person

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Remove arbitrary person $\rightarrow$ yield tournament on $n-1$ people.

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Theorem: All horses have the same color.

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As we will see, it is more subtle to catch errors in proofs of correct theorems!!

Strong Induction and Recursion.

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def find-x-y(n):
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   elif (n==13): return(2,1)
   elif (n==14): return(1,2)
   elif (n==15): return(0,3)
   else:
     (x',y') = find-x-y(n-4)
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Base cases:

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Strong Induction and Recursion.

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Slight differences: showed for all $n \geq 16$ that $\wedge_{i=4}^{n-1} P(i) \implies P(n)$.

Summary: principle of induction.

Today: More induction.

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Strengthen theorem statement.
Sum of first n odds is n^2.
Hole anywhere.
Not same as strong induction.

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