Today.

Couple of more induction proofs.
Stable Marriage.
Strengthening: need to...

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 \). (\( S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).)

Base: \( P(1) \). \( 1 \leq 2 \).

Ind Step: \( \sum_{i=1}^{k+1} \frac{1}{i^2} \leq 2 \).

\[
\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2}.
\]

\[
\leq 2 + \frac{1}{(k+1)^2}
\]

Uh oh?

Hmmm... It better be that any sum is *strictly less than* 2.

How much less? At least by \( \frac{1}{(k+1)^2} \) for \( S_k \).

“\( S_k \leq 2 - \frac{1}{(k+1)^2} \)” \( \implies \) “\( S_{k+1} \leq 2 \)”

Induction step works! No! Not the same statement!!!!

Need to prove “\( S_{k+1} \leq 2 - \frac{1}{(k+2)^2} \).

Darn!!!
**Strengthening: how?**

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n) \). (\( S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).)

**Proof:**

Ind hyp: \( P(k) \) — “\( S_k \leq 2 - f(k) \)”

Prove: \( P(k + 1) \) — “\( S_{k+1} \leq 2 - f(k + 1) \)”

\[
S(k + 1) = S_k + \frac{1}{(k+1)^2}
\]

\[
\leq 2 - f(k) + \frac{1}{(k+1)^2} \quad \text{By ind. hyp.}
\]

Choose \( f(k + 1) \leq f(k) - \frac{1}{(k+1)^2} \).

\[
\Rightarrow S(k + 1) \leq 2 - f(k + 1).
\]

Can you?

- Subtracting off a quadratically decreasing function every time.
- Maybe a linearly decreasing function to keep positive?

Try \( f(k) = \frac{1}{k} \)

\[
\frac{1}{k+1} \leq \frac{1}{k} - \frac{1}{(k+1)^2} \quad ?
\]

\[
1 \leq \frac{k+1}{k} - \frac{1}{k+1} \quad \text{Multiplied by } k + 1.
\]

\[
1 \leq 1 + \left( \frac{1}{k} - \frac{1}{k+1} \right) \quad \text{Some math. So yes!}
\]

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n} \).
Stable Marriage Problem

- Small town with $n$ boys and $n$ girls.
- Each girl has a ranked preference list of boys.
- Each boy has a ranked preference list of girls.

How should they be matched?
Count the ways..

- Maximize total satisfaction.
- Maximize number of first choices.
- Maximize worse off.
- Minimize difference between preference ranks.
The best laid plans..

Consider the couples..

- Jennifer and Brad
- Angelina and Billy-Bob

Brad prefers Angelina to Jennifer.
Angelina prefers Brad to BillyBob.
Uh..oh.
Produce a pairing where there is no running off!

**Definition:** A **pairing** is a disjoint set of $n$ boy-girl pairs.

Example: A pairing $S = \{(Brad, Jen); (BillyBob, Angelina)\}$.

**Definition:** A rogue couple $b, g^*$ for a pairing $S$: $b$ and $g^*$ prefer each other to their partners in $S$.

Example: Brad and Angelina are a rogue couple in $S$. 
A stable pairing??

Given a set of preferences.
Is there a stable pairing?
How does one find it?

Consider a single gender version: stable roommates.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td>B</td>
<td>C</td>
<td>D</td>
</tr>
<tr>
<td>B</td>
<td>C</td>
<td>A</td>
<td>D</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>A</td>
<td>B</td>
<td>D</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>A</td>
<td>B</td>
<td>C</td>
<td></td>
</tr>
</tbody>
</table>

Diagram:

- A is connected to B and D
- B is connected to A and C
- C is connected to A and D
- D is connected to A and C
The Traditional Marriage Algorithm.

Each Day:

1. Each boy **proposes** to his favorite girl on his list.
2. Each girl rejects all but her favorite proposer (whom she puts on a **string**.)
3. Rejected boy **crosses** rejecting girl off his list.

Stop when each girl gets exactly one proposal.
Does this terminate?

...produce a pairing?

....a stable pairing?

Do boys or girls do “better”? 

Example.

<table>
<thead>
<tr>
<th></th>
<th>Boys</th>
<th></th>
<th>Girls</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
<td>B</td>
<td>C</td>
<td>1</td>
</tr>
<tr>
<td>A</td>
<td>X</td>
<td>2</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>B</td>
<td>X</td>
<td>X</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
<td>X</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Day 1</th>
<th>Day 2</th>
<th>Day 3</th>
<th>Day 4</th>
<th>Day 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A, B</td>
<td>A</td>
<td>X, C</td>
<td>C</td>
<td>C</td>
</tr>
<tr>
<td>2</td>
<td>C</td>
<td>B, C</td>
<td>B</td>
<td>A, B</td>
<td>A</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>B</td>
</tr>
</tbody>
</table>
Termination.

Every non-terminated day a boy crossed an item off the list. Total size of lists? \( n \) boys, \( n \) length list. \( n^2 \) Terminates in at most \( n^2 + 1 \) steps!
It gets better every day for girls..

**Improvement Lemma:** It just gets better for girls.
If on day $t$ a girl, $g$, has a boy $b$ on a string, any boy, $b'$, on $g$'s string for any day $t' > t$ is at least as good as $b$.

**Proof:**
$P(k)$ - “boy on $g$’s string is at least as good as $b$ on day $t + k$”

$P(0)$ - true. Girl has $b$ on string.

Assume $P(k)$. Let $b'$ be boy on string on day $t + k$.

On day $t + k + 1$, boy $b'$ comes back.

Girl can choose $b'$, or do better with another boy, $b''$

That is, $b \leq b'$ by induction hypothesis.

And $b''$ is better than $b'$ by algorithm.

$P(k) \implies P(k + 1)$. And by principle of induction.
Lemma: Every boy is matched at end.

Proof: If not, a boy $b$ must have been rejected $n$ times.

Every girl has been proposed to by $b$, and Improvement lemma

$\implies$ each girl has a boy on a string.

and each boy on at most one string.

$n$ girls and $n$ boys. Same number of each.

$\implies b$ must be on some girl’s string!

Contradiction.
Pairing is Stable.

**Lemma:** There is no rogue couple for the pairing formed by traditional marriage algorithm.

**Proof:**
Assume there is a rogue couple; \((b, g^*)\)

\[
\begin{align*}
&b^* \overset{\text{likes}}{\longrightarrow} g^* \quad \text{\(b\) likes \(g^*\) more than \(g\).} \\
&b \overset{\text{likes}}{\longrightarrow} g \quad \text{\(g^*\) likes \(b\) more than \(b^*\).}
\end{align*}
\]

Boy \(b\) proposes to \(g^*\) before proposing to \(g\).
So \(g^*\) rejected \(b\) (since he moved on)
By improvement lemma, \(g^*\) likes \(b^*\) better than \(b\).
Contradiction!
Good for boys? girls?

Is the TMA better for boys? for girls?

**Definition:** A pairing is $x$-optimal if $x$’s partner is its best partner in any stable pairing.

**Definition:** A pairing is $x$-pessimal if $x$’s partner is its worst partner in any stable pairing.

**Definition:** A pairing is boy optimal if it is $x$-optimal for all boys $x$.

.. and so on for boy pessimal, girl optimal, girl pessimal.

**Claim:** The optimal partner for a boy must be first in his preference list.

True? False? False!

Subtlety here: Best partner in any stable pairing.

As well as you can in a globally stable solution!

**Question:** Is there a boy or girl optimal pairing?

Is it possible:

$b$-optimal pairing different from the $b'$-optimal pairing!

Yes? No?
TMA is optimal!

For boys? For girls?

**Theorem:** TMA produces a boy-optimal pairing.

**Proof:**
Assume not: there are boys who do not get their optimal girl.

Let $t$ be first day a boy $b$ gets rejected by his optimal girl $g$ who he is paired with in stable pairing $S$.

$b^* -$ knocks $b$ off of $g$'s string on day $t$ $\implies$ $g$ prefers $b^*$ to $b$

By choice of $t$, $b^*$ prefers $g$ to optimal girl.

$\implies b^*$ prefers $g$ to his partner $g^*$ in $S$.

Rogue couple for $S$. So $S$ is not a stable pairing. Contradiction.

Notes: $S$ - stable. $(b^*, g^*) \in S$. But $(b^*, g)$ is rogue couple!

Used Well-Ordering principle...Induction.
How about for girls?

**Theorem**: TMA produces girl-pessimal pairing.

*T* – pairing produced by TMA.

*S* – worse stable pairing for girl *g*.

In *T*, 

\[(g, b)\]

is pair.

In *S*, 

\[(g, b^*)\]

is pair.

*g* likes *b*\(^*\) less than she likes *b*.

*T* is boy optimal, so *b* likes *g* more than his partner in *S*.

\[(g, b)\]

is Rogue couple for *S*

*S* is not stable.

Contradiction.

Notes: Not really induction.

Structural statement: Boy optimality \(\implies\) Girl pessimality.
Quick Questions.

How does one make it better for girls?

- SMA - stable marriage algorithm. One side proposes.
- TMA - boys propose.
- Girls could propose. $\implies$ optimal for girls.
The method was used to match residents to hospitals.

Hospital optimal...

..until 1990’s...Resident optimal.

Variations: couples,
Don’t go!

Summary.