Today.

Couple of more induction proofs.

Stable Marriage.

Stable Marriage Problem

- Small town with \( n \) boys and \( n \) girls.
- Each girl has a ranked preference list of boys.
- Each boy has a ranked preference list of girls.

How should they be matched?

Count the ways..

- Maximize total satisfaction.
- Maximize number of first choices.
- Maximize worse off.
- Minimize difference between preference ranks.

The best laid plans..

Consider the couples..

- Jennifer and Brad
- Angelina and Billy-Bob

Brad prefers Angelina to Jennifer.

Angelina prefers Brad to Billy-Bob.

Uh...oh.

Strengthening: need to...

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 \). \( (S_n = \sum_{i=1}^{n} \frac{1}{i^2} ) \)

Base: \( P(1). 1 \leq 2. \)

Ind Step: \( \sum_{i=1}^{k+1} \frac{1}{i^2} \leq 2. \)

\[
\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \\
\leq 2 + \frac{1}{(k+1)^2} \\
\]

Uh oh??

Hmmm... It better be that any sum is strictly less than 2.

How much less? At least by \( \frac{1}{(k+1)^2} \) for \( S_k \).

\("S_k \leq 2 - \frac{1}{(k+1)^2} \) \( \Rightarrow \) "\( S_{k+1} \leq 2 " \)

Induction step works! No! Not the same statement!!!!

Need to prove "\( S_{k+1} \leq 2 - \frac{1}{(k+1)^2} \)."

Darn!!!

Strengthening: how?

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n). \) \( (S_n = \sum_{i=1}^{n} \frac{1}{i^2} ) \)

Proof:

Ind hyp: \( P(k) \) — "\( S_k \leq 2 - f(k) " \)

Prove: \( P(k+1) \) — "\( S_{k+1} \leq 2 - f(k+1) " \)

\[
S(k+1) = S_k + \frac{1}{(k+1)^2} \\
\leq 2 - f(k) + \frac{1}{(k+1)^2} \text{ By ind. hyp.} \\
\]

Choose \( f(k+1) = f(k) - \frac{1}{(k+1)^2} \).

\[
S(k+1) \leq 2 - f(k+1) \\
\]

Can you?

Subtracting off a quadratically decreasing function every time.

Maybe a linearly decreasing function to keep positive?

Try \( f(k) = \frac{k}{2} \)

\[
\frac{1}{k+1} \leq \frac{1}{2} - \frac{1}{(k+1)^2} ? \]

\[
1 \leq \frac{1}{2} - \frac{1}{(k+1)} \text{ Multiplied by } k+1. \\
1 \leq 1 + \left(\frac{1}{2} - \frac{1}{k+1}\right) \text{ Some math. So yes!} \\
\]

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n} \).

Some math.

So yes!

The best laid plans..

Consider the couples..

- Jennifer and Brad
- Angelina and Billy-Bob

Brad prefers Angelina to Jennifer.

Angelina prefers Brad to Billy-Bob.

Uh...oh.
So..

Produce a pairing where there is no running off!

**Definition:** A pairing is disjoint set of \( n \) boy-girl pairs.

Example: A pairing \( S = \{(\text{Brad}, \text{Jen}), (\text{BillyBob}, \text{Angelina})\} \).

**Definition:** A rogue couple \( b, g^* \) for a pairing \( S \):

- \( b \) and \( g^* \) prefer each other to their partners in \( S \)

Example: Brad and Angelina are a rogue couple in \( S \).

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**A stable pairing??**

Given a set of preferences.

Is there a stable pairing?

How does one find it?

Consider a single gender version: stable roommates.

A B C D
B X 3
C X 1 X
D 3 A C B

Every non-terminated day a boy crossed an item off the list.

Terminates in at most \( n^2 + 1 \) steps!

**The Traditional Marriage Algorithm.**

Each Day:

1. Each boy proposes to his favorite girl on his list.
2. Each girl rejects all but her favorite proposer (whom she puts on a string.)
3. Rejected boy crosses rejecting girl off his list.

Stop when each girl gets exactly one proposal.
Does this terminate?

...produce a pairing?

....a stable pairing?
Do boys or girls do "better"?

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**Termination.**

Every non-terminated day a boy crossed an item off the list.

Total size of lists? \( n \) boys, \( n \) length list. \( n^2 \)

Terminates in at most \( n^2 + 1 \) steps!
Pairing when done.

Lemma: Every boy is matched at end.
Proof: If not, a boy b must have been rejected n times.
Every girl has been proposed to by b, and Improvement lemma → each girl has a boy on a string.
and each boy on at most one string.
n girls and n boys. Same number of each.
→ b must be on some girl’s string!
Contradiction.

Pairing is Stable.

Lemma: There is no rogue couple for the pairing formed by traditional marriage algorithm.
Proof: Assume there is a rogue couple: (b, g*)

b' ------- g'  b likes g' more than g.

b ------- g  g' likes b more than b'.

Boy b proposes to g' before proposing to g.
So g' rejected b (since he moved on)
By improvement lemma, g' likes b' better than b.
Contradiction!

TMA is optimal!

For boys? For girls?
Theorem: TMA produces a boy-optimal pairing.
Proof: Assume not: there are boys who do not get their optimal girl.
Let t be first day a boy b gets rejected
by his optimal girl g who he is paired with in stable pairing S.
b' - knocks b off of g's string on day t → g prefers b' to b
By choice of t, b' prefers g to optimal girl.
→ b' prefers g to his partner g' in S.

Rogue couple for S.
So S is not a stable pairing. Contradiction.
Notes: S - stable. (b', g') ∈ S. But (b', g) is rogue couple!
Used Well-Ordering principle...Induction.

Good for boys? girls?

Is the TMA better for boys? for girls?
Definition: A pairing is x-optimal if x’s partner is its best partner in any stable pairing.
Definition: A pairing is x-pessimal if x’s partner is its worst partner in any stable pairing.
Definition: A pairing is boy optimal if it is x-optimal for all boys x.
...and so on for boy pessimal, girl optimal, girl pessimal.
Claim: The optimal partner for a boy must be first in his preference list.
True? False? False!
Subtlety here: Best partner in any stable pairing.
As well as you can in a globally stable solution!

Question: Is there a boy or girl optimal pairing?
Is it possible: b-optimal pairing different from the b'-optimal pairing!
Yes? No?

How about for girls?

Theorem: TMA produces girl-pessimal pairing.
T – pairing produced by TMA.
S – worse stable pairing for girl g.
In T, (g, b) is pair.
In S, (g, b') is pair.
g likes b' less than she likes b.
T is boy optimal, so b likes g more than his partner in S.
(g, b) is best partner for S.
S is not stable.
Contradiction.
Notes: Not really induction.
Structural statement: Boy optimality → Girl pessimality.

Quick Questions.

How does one make it better for girls?
SMA - stable marriage algorithm. One side proposes.
TMA - boys propose.
Girls could propose. → optimal for girls.
Residency Matching.

The method was used to match residents to hospitals.
Hospital optimal...
...until 1990's... Resident optimal.
Variations: couples.

Don't go!

Summary.