CS70 - Lecture 6

Graphs: Coloring; Special Graphs
1. Review of L5
Graphs: Coloring; Special Graphs

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2. Planar Five Color Theorem
Graphs: Coloring; Special Graphs

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3. Special Graphs:
Graphs: Coloring; Special Graphs

1. Review of L5
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3. Special Graphs:
   - Trees:
Graphs: Coloring; Special Graphs

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   ▶ Trees: Three characterizations
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   ▶ Trees: Three characterizations
   ▶ Hypercubes:
Graphs: Coloring; Special Graphs

1. Review of L5
2. Planar Five Color Theorem
3. Special Graphs:
   - Trees: Three characterizations
   - Hypercubes: Strongly connected!
You need to submit your grading option (HW or Test Only) by Thursday night at 10pm!

Instruction is at Piazza @241

If you don’t submit your response on time, the default will be the homework option.
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Administration

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Definitions: graph, walk, tour, path, cycle, Eulerian tour

There exists an Eulerian Tour if and only if the graph is connected and every vertex has even degree.

Only if: If a vertex $v$ has odd degree, you will get stuck there.

This solves the Konigsberg problem.

If: Induction on $e$ (number of edges).

Euler Formula: Planar + Connected $\Rightarrow v + f = e + 2$.

Proof: Induction on $e$.

- Planar $\Rightarrow 2e \geq 3f \Rightarrow 3v \geq e + 6 \Rightarrow K_5$ is non-planar

- Planar + Bipartite $\Rightarrow 2e \geq 4f \Rightarrow 2v \geq e + 4 \Rightarrow K_3, 3$ is non-planar
Review of L5

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Planar $\Rightarrow 2e \geq 3f$
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Review of L5: Preliminaries

- Was Euler alive before or after Euclid?
  - After! Euler = 1707–1783.
  - Euclid ≈ 300BC

- Was Euler alive before or after Newton?
  - After! Newton = 1642–1726.

- What was Euler’s first name?
  - Leonhard.

- Was Euler a freak of nature?
  - Definitely!

- $e^{i\pi} = -1$, graphs, number theory, physics, astronomy, more than 800 papers, ...
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Review of L5: Q1

Question: What is this argument?

Proof by induction on $e$ of the existence of a Eulerian tour in a connected even-degrees graph. There is one in $G$, so that there is one in the original graph.
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What is this argument?

A proof by induction on $e$ of Euler's formula: $v + f = e + 2$. 
What is this argument?
What is this argument?

e edges
f faces
v vertices

$e' = e - 1$
$f' = f - 1$
$v' = v$
What is this argument?

A proof by induction on $e$ of Euler’s formula: $v + f = e + 2$. 
Review of L5: Q3

A proof that $K_5$ is non-planar.

Where does (*) come from?
Every cycle has at least 3 edges.

Where does (**) come from?
Euler's formula.

Let's remember: Planar $\Rightarrow e \leq 3v - 6$
What is this argument?
What is this argument?

\[ 2e = |E/F \text{ adjacencies} | \geq 3f \quad (*) \]
\[ 3v + 3f = 3e + 6 \quad (**) \]
\[ 3v + 2e \geq 3e + 6 \]
\[ 3v \geq e + 6 \]

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$3v + 2e \geq 3e + 6$  

$3v \geq e + 6$

$v = 5, e = 10$
What is this argument?

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Let’s remember: $\text{Planar} \implies e \leq 3v - 6$
A proof that \( K_{3,3} \) is non-planar.

Where does (*) come from?

Every cycle has at least 4 edges.

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Euler's formula.
Review of L5: Q3

What is this argument?

2v + 2f = 2e + 4
2v + e  2e +4
2v  e +4
2e = |E/F adjacencies |  4f
2 f

K 3 , 3

Bipartite

A proof that $K_3, 3$ is non-planar.

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Euler's formula.
What is this argument?

Bipartite
\[ 2e = |E/F \text{ adjacencies}| \geq 4f \quad (*) \]
\[ e \geq 2f \]
\[ 2v + 2f = 2e + 4 \quad (**') \]
\[ 2v + e \geq 2e + 4 \]
\[ 2v \geq e + 4 \]

\[ K_{3,3} \]

\[ e = 9, v = 6 \]
What is this argument?

Bipartite

\[ 2e = |E/F \text{ adjacencies}| \geq 4f \quad (*) \]
\[ e \geq 2f \]
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A proof that \( K_{3,3} \) is non-planar.
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$2e = |E/F \text{ adjacencies}| \geq 4f$ (*)

$e \geq 2f$

$2v + 2f = 2e + 4$ (**)

$2v + e \geq 2e + 4$

$2v \geq e + 4$

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\[ 2v + e \geq 2e + 4 \]

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\begin{align*}
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e &\geq 2f \\
2v + 2f &= 2e + 4 \quad (**) \\
2v + e &\geq 2e + 4 \\
2v &\geq e + 4
\end{align*}
What is this argument?

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\[ 2e = |E/F \text{ adjacencies} \mid \geq 4f \quad (*) \]
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\[ 2v + 2f = 2e + 4 \quad (**), \]
\[ 2v + e \geq 2e + 4 \]
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A proof that \( K_{3,3} \) is non-planar.

Where does \((*)\) come from? Every cycle has at least 4 edges.

Where does \((**)\) come from? Euler’s formula.
Graph Coloring.

Given $G = (V, E)$, a coloring of a $G$ assigns colors to vertices $V$ where for each edge the endpoints have different colors.
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![Graph Coloring Example](image)
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Notice that the last graph has one three-color coloring.
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Notice that the last graph has one three-color coloring.  
→ Fewer colors than number of vertices.
Graph Coloring.

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Notice that the last graph has one three-color coloring.

→ Fewer colors than number of vertices.
→ Fewer colors than the maximum degree of the nodes.
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Interesting things to do.
Graph Coloring.

Given $G = (V, E)$, a coloring of a $G$ assigns colors to vertices $V$ where for each edge the endpoints have different colors.

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Interesting things to do. Algorithm!
Planar graphs and maps.

Planar graph coloring $\equiv$ map coloring.

Four color theorem is about planar graphs! It says that every planar graph (or map) can be colored with four colors!

Stated in 1852. Proved in 1976 ... by reducing the problem to 1936 cases (400 pages of analysis) and checking these cases by computer!
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Six color theorem.

**Theorem:** Every planar graph can be colored with six colors.
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**Proof:**
Six color theorem.

**Theorem:** Every planar graph can be colored with six colors.

**Proof:**
Induction on \( v \).
Theorem: Every planar graph can be colored with six colors.

Proof:
Induction on $v$.

Recall: $e \leq 3v - 6$ for any planar graph.
**Theorem:** Every planar graph can be colored with six colors.

**Proof:**
Induction on $v$.

Recall: $e \leq 3v - 6$ for any planar graph.

Total degree (sum of the degrees): $2e$
Six color theorem.

**Theorem:** Every planar graph can be colored with six colors.

**Proof:**
Induction on $v$.

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Total degree (sum of the degrees): $2e$

Average degree: $\frac{2e}{v}$
**Theorem:** Every planar graph can be colored with six colors.

**Proof:**

Induction on $v$.

Recall: $e \leq 3v - 6$ for any planar graph.

Total degree (sum of the degrees): $2e$

Average degree: $\frac{2e}{v} \leq \frac{2(3v-6)}{v}$
**Theorem:** Every planar graph can be colored with six colors.

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Total degree (sum of the degrees): $2e$

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**Theorem:** Every planar graph can be colored with six colors.

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$\Rightarrow$ There exists a vertex $x$ with degree $< 6$.
Theorem: Every planar graph can be colored with six colors.

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Induction on $v$.

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Total degree (sum of the degrees): $2e$

Average degree: $\frac{2e}{v} \leq \frac{2(3v-6)}{v} \leq 6 - \frac{12}{v} < 6$.

$\Rightarrow$ There exists a vertex $x$ with degree $< 6$ or at most 5.
Six color theorem.

**Theorem:** Every planar graph can be colored with six colors.

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- Remove vertex $x$ of degree at most 5.
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$\Rightarrow$ There exists a vertex $x$ with degree $< 6$ or at most 5.

Remove vertex $x$ of degree at most 5.

Inductively color remaining graph with the six colors.
Theorem: Every planar graph can be colored with six colors.

Proof:
Induction on $v$.

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One of the six colors is available for $x$ since only five neighbors...
Theorem: Every planar graph can be colored with six colors.

Proof:
Induction on \( v \).

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Total degree (sum of the degrees): \( 2e \)
Average degree: \( \frac{2e}{v} \leq \frac{2(3v-6)}{v} \leq 6 - \frac{12}{v} < 6. \)

\( \Rightarrow \) There exists a vertex \( x \) with degree \(< 6 \) or at most 5.

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One of the six colors is available for \( x \) since only five neighbors...
Six color theorem.

**Theorem**: Every planar graph can be colored with **six** colors.

A picture of the proof by induction:
Six color theorem.

**Theorem:** Every planar graph can be colored with six colors.

A picture of the proof by induction:
Six color theorem.

**Theorem:** Every planar graph can be colored with six colors.

A picture of the proof by induction:
Five color theorem

Theorem: Every planar graph can be colored with five colors.
Five color theorem

Theorem: Every planar graph can be colored with five colors.

Proof:
Five color theorem

Theorem: Every planar graph can be colored with five colors.

Proof:
Preliminary Observation: You can switch two colors in a legal coloring.
Five color theorem

Theorem: Every planar graph can be colored with five colors.

Proof:
Preliminary Observation: You can switch two colors in a legal coloring. Obvious!
Five color theorem

Theorem: Every planar graph can be colored with five colors.

Proof:
Preliminary Observation: You can switch two colors in a legal coloring. Obvious!

Consider again the degree 5 vertex.
Five color theorem

Theorem: Every planar graph can be colored with five colors.

Proof:
Preliminary Observation: You can switch two colors in a legal coloring. Obvious!

Consider again the degree 5 vertex. Again recurse: Assume five colors.
Five color theorem

Theorem: Every planar graph can be colored with five colors.

Proof:
Preliminary Observation: You can switch two colors in a legal coloring. Obvious!

Consider again the degree 5 vertex. Again recurse: Assume five colors.

Assume neighbors are colored all differently.

Assume neighbors are colored all differently.

\[
\begin{array}{c}
\text{Red} \quad \text{Red} \\
\text{Blue} \quad \text{Red} \\
\text{Green} \quad \text{Red} \\
\text{Orange} \quad \text{Red} \\
\end{array}
\]

What color is it?
Must be blue or green to be on that path.
Must be red or orange to be on that path.
Contradiction.
Can recolor one of the neighbors.
And recolor “center” vertex.
Five color theorem

Theorem: Every planar graph can be colored with five colors.

Proof:

Preliminary Observation: You can switch two colors in a legal coloring. Obvious!

Consider again the degree 5 vertex. Again recurse: Assume five colors.

Assume neighbors are colored all differently. Otherwise done.

Switch green to blue in component. Done.

Unless blue-green path to blue.

Switch red to orange in its component. Done.

Unless red-orange path to red.

Planar. ⇒ paths intersect at a vertex! What color is it?

Must be blue or green to be on that path.

Must be red or orange to be on that path.

Contradiction.

Can recolor one of the neighbors. And recolor "center" vertex.
Five color theorem

Theorem: Every planar graph can be colored with five colors.

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Consider again the degree 5 vertex. Again recurse: Assume five colors.

Assume neighbors are colored all differently. Otherwise done.

Switch green to blue in component.

Planar. =⇒ paths intersect at a vertex!

What color is it? Must be blue or green to be on that path.

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Theorem: Every planar graph can be colored with five colors.

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Consider again the degree 5 vertex. Again recurse: Assume five colors.

Assume neighbors are colored all differently. Otherwise done.
Switch green to blue in component. Done.

What color is it? Must be blue or green to be on that path. Must be red or orange to be on that path. Contradiction. Can recolor one of the neighbors and recolor “center” vertex.
**Five color theorem**

Theorem: Every planar graph can be colored with **five** colors.

**Proof:**

Preliminary Observation: You can switch two colors in a legal coloring. Obvious!

Consider again the degree 5 vertex. Again recurse: Assume five colors.

Assume neighbors are colored all differently. Otherwise done.

Switch green to blue in component. Done. Unless **blue-green** path to blue.
Five color theorem

Theorem: Every planar graph can be colored with five colors.

Proof:
Preliminary Observation: You can switch two colors in a legal coloring. Obvious!

Consider again the degree 5 vertex. Again recurse: Assume five colors.

Assume neighbors are colored all differently. Otherwise done.
Switch green to blue in component.
Done. Unless blue-green path to blue.

Planar. ⇒ paths intersect at a vertex!
What color is it?
Must be blue or green to be on that path.
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Consider again the degree 5 vertex. Again recurse: Assume five colors.

Assume neighbors are colored all differently. Otherwise done.

Switch green to blue in component.
Done. Unless blue-green path to blue.
Switch red to orange in its component.
Five color theorem

Theorem: Every planar graph can be colored with five colors.

Proof:
Preliminary Observation: You can switch two colors in a legal coloring. Obvious!

Consider again the degree 5 vertex. Again recurse: Assume five colors.

Assume neighbors are colored all differently.
Otherwise done.
Switch green to blue in component.
Done. Unless blue-green path to blue.
Switch red to orange in its component.
Done.

...
Five color theorem

Theorem: Every planar graph can be colored with five colors.

Proof:

Preliminary Observation: You can switch two colors in a legal coloring. Obvious!

Consider again the degree 5 vertex. Again recurse: Assume five colors.

Assume neighbors are colored all differently. Otherwise done.

Switch green to blue in component. Done. Unless blue-green path to blue.

Switch red to orange in its component. Done. Unless red-orange path to red.
Five color theorem

Theorem: Every planar graph can be colored with **five** colors.

Proof:

Preliminary Observation: You can switch two colors in a legal coloring. Obvious!

Consider again the degree 5 vertex. Again recurse: Assume five colors.

Assume neighbors are colored all differently.
Otherwise done.
Switch green to blue in component.
Done. Unless **blue-green** path to blue.
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Theorem: Every planar graph can be colored with five colors.

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Planar.
Five color theorem

Theorem: Every planar graph can be colored with five colors.

Proof:

Preliminary Observation: You can switch two colors in a legal coloring. Obvious!

Consider again the degree 5 vertex. Again recurse: Assume five colors.

Assume neighbors are colored all differently. Otherwise done.
Switch green to blue in component.
Done. Unless blue-green path to blue.
Switch red to orange in its component.
Done. Unless red-orange path to red.
Planar. \[\implies\] paths intersect at a vertex!
Theorem: Every planar graph can be colored with five colors.

Proof:

Preliminary Observation: You can switch two colors in a legal coloring. Obvious!

Consider again the degree 5 vertex. Again recurse: Assume five colors.

Assume neighbors are colored all differently. Otherwise done.

Switch green to blue in component.

Done. Unless blue-green path to blue.

Switch red to orange in its component.

Done. Unless red-orange path to red.

Planar. $\implies$ paths intersect at a vertex!

What color is it?
Five color theorem

Theorem: Every planar graph can be colored with five colors.

Proof:
Preliminary Observation: You can switch two colors in a legal coloring. Obvious!

Consider again the degree 5 vertex. Again recurse: Assume five colors.

Assume neighbors are colored all differently. Otherwise done.
Switch green to blue in component.
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Planar. \( \Rightarrow \) paths intersect at a vertex!

What color is it?
Must be blue or green to be on that path.
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Consider again the degree 5 vertex. Again recurse: Assume five colors.

Assume neighbors are colored all differently. Otherwise done.
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Consider again the degree 5 vertex. Again recurse: Assume five colors.

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Switch green to blue in component. Done. Unless blue-green path to blue.

Switch red to orange in its component. Done. Unless red-orange path to red.

Planar. $\implies$ paths intersect at a vertex!

What color is it?

Must be blue or green to be on that path.

Must be red or orange to be on that path.

Contradiction.
Five color theorem

Theorem: Every planar graph can be colored with five colors.

Proof:
Preliminary Observation: You can switch two colors in a legal coloring. Obvious!

Consider again the degree 5 vertex. Again recurse: Assume five colors.

Assume neighbors are colored all differently. Otherwise done.

Switch green to blue in component.
Done. Unless blue-green path to blue.

Switch red to orange in its component.
Done. Unless red-orange path to red.

Planar. \( \implies \) paths intersect at a vertex!

What color is it?
Must be blue or green to be on that path.
Must be red or orange to be on that path.

Contradiction. Can recolor one of the neighbors. And recolor “center” vertex.
Five color theorem

Theorem: Every planar graph can be colored with five colors.

Proof:

Preliminary Observation: You can switch two colors in a legal coloring. Obvious!

Consider again the degree 5 vertex. Again recurse: Assume five colors.

Assume neighbors are colored all differently. Otherwise done.

Switch green to blue in component.
Done. Unless blue-green path to blue.

Switch red to orange in its component.
Done. Unless red-orange path to red.

Planar. \[\Rightarrow\] paths intersect at a vertex!

What color is it?
Must be blue or green to be on that path.
Must be red or orange to be on that path.

Contradiction. Can recolor one of the neighbors. And recolor “center” vertex.
Four Color Theorem

Theorem:
Any planar graph can be colored with four colors.

Proof:
Not Today!
Theorem: Any planar graph can be colored with four colors.
Theorem: Any planar graph can be colored with four colors.

Proof:
Four Color Theorem

**Theorem:** Any planar graph can be colored with four colors.

**Proof:** Not Today!
**Theorem:** Any planar graph can be colored with four colors.

**Proof:** Not Today!
A Tree, a tree.

Graph $G = (V, E)$.
Binary Tree!

More generally.
Trees.

Definitions:

1. An equivalent graph without a cycle.
2. A connected graph with $|V| - 1$ edges.
3. A connected graph where any edge removal disconnects it.
Trees.

Definitions: (Equivalent, as we prove later)
Trees.

Definitions: (Equivalent, as we prove later)

A connected graph without a cycle.
Trees.

Definitions: (Equivalent, as we prove later)

A connected graph without a cycle.
A connected graph with \(|V| - 1\) edges.
Trees.

Definitions: (Equivalent, as we prove later)

- A connected graph without a cycle.
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Trees.

Definitions: (Equivalent, as we prove later)

A connected graph without a cycle.
A connected graph with $|V| - 1$ edges.
A connected graph where any edge removal disconnects it.

Some trees.

No cycle and connected?
Trees.

Definitions: (Equivalent, as we prove later)

A connected graph without a cycle.
A connected graph with $|V| - 1$ edges.
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Some trees.

No cycle and connected? Yes.
Trees.

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No cycle and connected? Yes.
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Removing any edge disconnects it.
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No cycle and connected? Yes.
$|V| - 1$ edges and connected? Yes.
Removing any edge disconnects it. Harder to check,
Trees.

Definitions: (Equivalent, as we prove later)

- A connected graph without a cycle.
- A connected graph with $|V| - 1$ edges.
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Some trees.

- No cycle and connected? Yes.
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- Removing any edge disconnects it. Harder to check, but yes.
Trees.

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- A connected graph without a cycle.
- A connected graph with $|V| - 1$ edges.
- A connected graph where any edge removal disconnects it.

Some trees.

No cycle and connected? Yes.
$|V| - 1$ edges and connected? Yes.
Removing any edge disconnects it. Harder to check, but yes.

To tree or not to tree!
Equivalence of Definitions.

**Theorem** These properties of a graph are equivalent:

1. A connected graph without a cycle.
2. A connected graph with $|V| - 1$ edges.
3. A connected graph where any edge removal disconnects it.

Proof:

1. $\Rightarrow$ 2.
   - Assume true for $|V| \leq n - 1$.
   - Consider $G$ with $|V| = n$.
   - Why is there an unpopular vertex with degree only 1?
   - Otherwise: cycle (enter-leave-enter-leave-enter).

$\Rightarrow G$ has $n - 1$ edges.
Theorem These properties of a graph are equivalent: (i.e., each implies any other)

1. A connected graph without a cycle.
2. A connected graph with \(|V| - 1\) edges.
3. A connected graph where any edge removal disconnects it.
Equivalence of Definitions.

**Theorem** These properties of a graph are equivalent: (i.e., each implies any other)

1. A connected graph without a cycle.
2. A connected graph with $|V| - 1$ edges.
3. A connected graph where any edge removal disconnects it.

**Proof:** (1) $\Rightarrow$ (2)
Theorem These properties of a graph are equivalent: (i.e., each implies any other)

1. A connected graph without a cycle.
2. A connected graph with $|V| - 1$ edges.
3. A connected graph where any edge removal disconnects it.

Proof: (1) $\Rightarrow$ (2)
Assume true for $|V| \leq n - 1$. 
Equivalence of Definitions.

**Theorem** These properties of a graph are equivalent: (i.e., each implies any other)

1. A connected graph without a cycle.
2. A connected graph with $|V| - 1$ edges.
3. A connected graph where any edge removal disconnects it.

**Proof:** (1) $\Rightarrow$ (2)
Assume true for $|V| \leq n - 1$. Consider $G$ with $|V| = n$. 
Theorem These properties of a graph are equivalent: (i.e., each implies any other)

(1) A connected graph without a cycle.
(2) A connected graph with $|V| - 1$ edges.
(3) A connected graph where any edge removal disconnects it.

Proof: $(1) \Rightarrow (2)$

Assume true for $|V| \leq n - 1$. Consider $G$ with $|V| = n$. 

![Diagram of a graph with $n-1$ vertices and $n-2$ edges, labeled as follows: v connected to the graph, which is disconnected when v is removed.](image.png)
Equivalence of Definitions.

**Theorem** These properties of a graph are equivalent: (i.e., each implies any other)

1. A connected graph without a cycle.
2. A connected graph with $|V| - 1$ edges.
3. A connected graph where any edge removal disconnects it.

**Proof:** (1) $\Rightarrow$ (2)
Assume true for $|V| \leq n - 1$. Consider $G$ with $|V| = n$.

Why is there an unpopular $v$ with degree only 1?
Equivalence of Definitions.

**Theorem** These properties of a graph are equivalent: (i.e., each implies any other)

1. A connected graph without a cycle.
2. A connected graph with $|V| - 1$ edges.
3. A connected graph where any edge removal disconnects it.

**Proof:** (1) $\Rightarrow$ (2)

Assume true for $|V| \leq n - 1$. Consider $G$ with $|V| = n$.

Why is there a unpopular $v$ with degree only 1? Otherwise: cycle
Theorem These properties of a graph are equivalent: (i.e., each implies any other)

1. A connected graph without a cycle.
2. A connected graph with $|V| - 1$ edges.
3. A connected graph where any edge removal disconnects it.

Proof: (1) $\Rightarrow$ (2)
Assume true for $|V| \leq n - 1$. Consider $G$ with $|V| = n$.

Why is there a unpopular $v$ with degree only 1? Otherwise: cycle (enter-leave-enter-leave-enter).
Theorem These properties of a graph are equivalent: (i.e., each implies any other)

1. A connected graph without a cycle.
2. A connected graph with \(|V| - 1\) edges.
3. A connected graph where any edge removal disconnects it.

Proof: (1) ⇒ (2)
Assume true for \(|V| \leq n - 1\). Consider \(G\) with \(|V| = n\).

Why is there a unpopular \(v\) with degree only 1? Otherwise: cycle (enter-leave-enter-leave-enter). \(\Rightarrow G\) has \(n - 1\) edges.
Equivalence of Definitions.

**Theorem** These properties of a graph are equivalent: (i.e., each implies any other)

1. A connected graph without a cycle.
2. A connected graph with $|V| - 1$ edges.
3. A connected graph where any edge removal disconnects it.

Proof:

Assume true for $|V| \leq n - 1$. Consider a graph $G$ with $n$ vertices and $n - 1$ edges. There must be some vertex with degree 1. Otherwise, the sum of degrees is at least $2n$. But the sum of degrees is $2|E| = 2(n - 1)$. Remove that vertex. Get a connected graph $G'$ without a cycle. Same for $G'$. 
Equivalence of Definitions.

**Theorem** These properties of a graph are equivalent: (i.e., each implies any other)

1. A connected graph without a cycle.
2. A connected graph with \(|V| - 1\) edges.
3. A connected graph where any edge removal disconnects it.

**Proof:** (2) \(\Rightarrow\) (1)
Equivalence of Definitions.

**Theorem** These properties of a graph are equivalent: (i.e., each implies any other)

1. A connected graph without a cycle.
2. A connected graph with $|V| - 1$ edges.
3. A connected graph where any edge removal disconnects it.

**Proof:** $(2) \implies (1)$

Assume true for $|V| \leq n - 1$.
Equivalence of Definitions.

**Theorem** These properties of a graph are equivalent: (i.e., each implies any other)

1. A connected graph without a cycle.
2. A connected graph with \(|V| - 1\) edges.
3. A connected graph where any edge removal disconnects it.

**Proof:** (2) \(\Rightarrow\) (1)
Assume true for \(|V| \leq n - 1\). Consider \(G\) connected with \(|V| = n\) vertices and \(n - 1\) edges.
**Theorem** These properties of a graph are equivalent: (i.e., each implies any other)

1. A connected graph without a cycle.
2. A connected graph with $|V| - 1$ edges.
3. A connected graph where any edge removal disconnects it.

**Proof:** $(2) \Rightarrow (1)$

Assume true for $|V| \leq n - 1$. Consider $G$ connected with $|V| = n$ vertices and $n - 1$ edges.

![Diagram of a graph with $n - 1$ vertices and $n - 2$ edges, with a vertex labeled v.]
Equivalence of Definitions.

**Theorem** These properties of a graph are equivalent: (i.e., each implies any other)

1. A connected graph without a cycle.
2. A connected graph with $|V| - 1$ edges.
3. A connected graph where any edge removal disconnects it.

**Proof:** (2) $\Rightarrow$ (1)

Assume true for $|V| \leq n - 1$. Consider $G$ connected with $|V| = n$ vertices and $n - 1$ edges.

$\Rightarrow$ There must be some $v$ with degree 1.
Theorem These properties of a graph are equivalent: (i.e., each implies any other)
(1) A connected graph without a cycle.
(2) A connected graph with $|V| - 1$ edges.
(3) A connected graph where any edge removal disconnects it.

Proof: (2) $\Rightarrow$ (1)
Assume true for $|V| \leq n - 1$. Consider $G$ connected with $|V| = n$ vertices and $n - 1$ edges.

$\Rightarrow$ There must be some $v$ with degree 1. Otherwise, sum of degrees $\geq 2n$. 
Equivalence of Definitions.

**Theorem** These properties of a graph are equivalent: (i.e., each implies any other)

1. A connected graph without a cycle.
2. A connected graph with $|V| - 1$ edges.
3. A connected graph where any edge removal disconnects it.

**Proof:** $(2) \Rightarrow (1)$
Assume true for $|V| \leq n - 1$. Consider $G$ connected with $|V| = n$ vertices and $n - 1$ edges.

⇒ There must be some $v$ with degree 1. Otherwise, sum of degrees $\geq 2n$. But sum of degrees $= 2|E| = 2(n - 1)$. 
Theorem These properties of a graph are equivalent: (i.e., each implies any other)

(1) A connected graph without a cycle.
(2) A connected graph with \(|V| - 1\) edges.
(3) A connected graph where any edge removal disconnects it.

Proof: (2) \(\Rightarrow\) (1)
Assume true for \(|V| \leq n - 1\). Consider \(G\) connected with \(|V| = n\) vertices and \(n - 1\) edges.

\(\Rightarrow\) There must be some \(v\) with degree 1. Otherwise, sum of degrees \(\geq 2n\). But sum of degrees \(= 2|E| = 2(n - 1)\).
Remove \(v\).
Equivalence of Definitions.

**Theorem** These properties of a graph are equivalent: (i.e., each implies any other)

1. A connected graph without a cycle.
2. A connected graph with $|V| - 1$ edges.
3. A connected graph where any edge removal disconnects it.

**Proof:** (2) $\Rightarrow$ (1)
Assume true for $|V| \leq n - 1$. Consider $G$ connected with $|V| = n$ vertices and $n - 1$ edges.

$\Rightarrow$ There must be some $v$ with degree 1. Otherwise, sum of degrees $\geq 2n$. But sum of degrees $= 2|E| = 2(n - 1)$.

Remove $v$. Get connected graph $G'$ without a cycle.
Equivalence of Definitions.

**Theorem** These properties of a graph are equivalent: (i.e., each implies any other)

1. A connected graph without a cycle.
2. A connected graph with $|V| - 1$ edges.
3. A connected graph where any edge removal disconnects it.

**Proof:** $(2) \Rightarrow (1)$
Assume true for $|V| \leq n - 1$. Consider $G$ connected with $|V| = n$ vertices and $n - 1$ edges.

$\Rightarrow$ There must be some $v$ with degree 1. Otherwise, sum of degrees $\geq 2n$. But sum of degrees $= 2|E| = 2(n - 1)$.

Remove $v$. Get connected graph $G'$ without a cycle. Same for $G$. \square
Theorem These properties of a graph are equivalent: (i.e., each implies any other)

1. A connected graph without a cycle.
2. A connected graph with $|V| - 1$ edges.
3. A connected graph where any edge removal disconnects it.

Proof:

1. $\Rightarrow$ 3

Assume true for $|V| \leq n - 1$.
Consider $G$ with $|V| = n$ and (1).
There is some $v$ with degree 1. (Otherwise, there is a cycle.)

If you remove the edge of $v$, you disconnect $G$.
If you remove any other edge, you disconnect $G'$, by induction hypothesis.
Theorem These properties of a graph are equivalent: (i.e., each implies any other)
(1) A connected graph without a cycle.
(2) A connected graph with $|V| - 1$ edges.
(3) A connected graph where any edge removal disconnects it.
Proof: (1) $\Rightarrow$ (3)
Theorem These properties of a graph are equivalent: (i.e., each implies any other)
(1) A connected graph without a cycle.
(2) A connected graph with $|V| - 1$ edges.
(3) A connected graph where any edge removal disconnects it.

Proof: (1) $\Rightarrow$ (3)
Assume true for $|V| \leq n - 1$. 
Equivalence of Definitions.

**Theorem** These properties of a graph are equivalent: (i.e., each implies any other)

1. A connected graph without a cycle.
2. A connected graph with $|V| - 1$ edges.
3. A connected graph where any edge removal disconnects it.

**Proof:** (1) $\Rightarrow$ (3)
Assume true for $|V| \leq n - 1$. Consider $G$ with $|V| = n$ and (1).
Theorem These properties of a graph are equivalent: (i.e., each implies any other)

(1) A connected graph without a cycle.
(2) A connected graph with $|V| - 1$ edges.
(3) A connected graph where any edge removal disconnects it.

Proof: (1) $\Rightarrow$ (3)
Assume true for $|V| \leq n - 1$. Consider $G$ with $|V| = n$ and (1).

There is some $v$ with degree 1.
Theorem These properties of a graph are equivalent: (i.e., each implies any other)

1. A connected graph without a cycle.
2. A connected graph with $|V| - 1$ edges.
3. A connected graph where any edge removal disconnects it.

Proof: (1) $\Rightarrow$ (3)

Assume true for $|V| \leq n - 1$. Consider $G$ with $|V| = n$ and (1).

There is some $v$ with degree 1. (Otherwise, there is a cycle.)
Theorem These properties of a graph are equivalent: (i.e., each implies any other)

1. A connected graph without a cycle.
2. A connected graph with $|V| - 1$ edges.
3. A connected graph where any edge removal disconnects it.

Proof: (1) $\Rightarrow$ (3)
Assume true for $|V| \leq n - 1$. Consider $G$ with $|V| = n$ and (1).

There is some $v$ with degree 1. (Otherwise, there is a cycle.)
Theorem These properties of a graph are equivalent: (i.e., each implies any other)

1. A connected graph without a cycle.
2. A connected graph with $|V| - 1$ edges.
3. A connected graph where any edge removal disconnects it.

Proof: (1) $\Rightarrow$ (3)
Assume true for $|V| \leq n - 1$. Consider $G$ with $|V| = n$ and (1). There is some $v$ with degree 1. (Otherwise, there is a cycle.)

If you remove the edge of $v$, you disconnect $G$. 
Equivalence of Definitions.

**Theorem** These properties of a graph are equivalent: (i.e., each implies any other)

1. A connected graph without a cycle.
2. A connected graph with \(|V| - 1\) edges.
3. A connected graph where any edge removal disconnects it.

**Proof:** \(1 \Rightarrow 3\)
Assume true for \(|V| \leq n - 1\). Consider \(G\) with \(|V| = n\) and (1).
There is some \(v\) with degree 1. (Otherwise, there is a cycle.)

If you remove the edge of \(v\), you disconnect \(G\). If you remove any other edge, you disconnect \(G'\), by induction hypothesis.
Equivalence of Definitions.

**Theorem** These properties of a graph are equivalent: (i.e., each implies any other)

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**Proof:** (3) $\Rightarrow$ (1)

Assume true for $|V| \leq n - 1$.
Consider $G$ with $|V| = n$ and (3).
There is some $v$ with degree 1.
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![Graph Diagram](image)
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Hypercubes.

Complete graphs, really connected!
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Complete graphs, really connected! But lots of edges.

$|V|(|V| - 1)/2$
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\[ |V|(|V| - 1)/2 \]

Trees,

\[ C_n \]

2^n vertices.

\[ 2^n \]

number of \( n \)-bit strings!

\[ n^2 \]

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2^n vertices each of degree \( n \)

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2^n vertices. number of n-bit strings!

\( n2^{n-1} \) edges.

\[
\begin{array}{c}
0 & 1 \\
\end{array}
\]

\[
\begin{array}{c}
00 & 01 & 10 & 11 \\
\end{array}
\]

\[
\begin{array}{c}
000 & 001 & 010 & 011 \\
100 & 101 & 110 & 111 \\
\end{array}
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A 0-dimensional hypercube is a node labelled with the empty string of bits.
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An $n$-dimensional hypercube consists of a 0-subcube (1-subcube) which is a $n-1$-dimensional hypercube with nodes labelled $0x$ ($1x$) with the additional edges ($0x, 1x$).
Recursive Definition.

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Cuts

Take a connected graph $G = (V, E)$ and some set $S \subseteq V$.

The cut $C$ is the set of edges that attach $S$ to $V - S$. 
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$S = \text{red nodes}$

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Hypercube: Can’t cut me!

Thm: In a hypercube, $|C| \geq \min\{|S|, |V - S|\}$. 

Examples:
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**Examples:**

```
|S| = 2  |S| = 1  |S| = 3  |S| = 2  
|C| = 2  |C| = 2  |C| = 2  |C| = 4  
```
Proof of Large Cuts.

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**Proof:**
Induction on $n$. 

\[
0 \quad 1 \\
0 \quad 1
\]

$S = \{0\}, |\text{cut edges}| = 1$

$S = \{\}, |\text{cut edges}| = 0$
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Induction Step Idea

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Here $S_0$ be the part of $S$ in left cube, $S_1$ in right-cube. Red edges are cut in each half-cube and blue edges across.
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Yes/No Computer Programs \( \equiv \) Boolean function on \( \{0, 1\}^n \)
The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on $\{0, 1\}^n$. Central area of study in computer science!

Yes/No Computer Programs $\equiv$ Boolean function on $\{0, 1\}^n$ Central object of study.
Summary of L6

Graphs: Coloring; Special Graphs

1. Review of L5
- Eulerian Tour iff connected and even degrees
- Euler Formula,
- $K_5$ and $K_{3,3}$ are non-planar

2. Planar Five Color Theorem

3. Special Graphs:
   - Trees: Three characterizations
   - Hypercubes: Strongly connected!

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     - Any cut $(S, V - S)$ has at least $\min\{|S|, |V - S|\}$ edges

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Have a nice weekend!