Today

Review for Midterm.
First there was logic...

A statement is a true or false.
First there was logic...

A statement is a true or false. Statements?
First there was logic...

A statement is a true or false.
Statements?

3 = 4 − 1 ?
First there was logic...

A statement is a true or false.
Statements?
  \(3 = 4 - 1\) ? Statement!
First there was logic...

A statement is a true or false.
Statements?

3 = 4 – 1 ? Statement!
3 = 5 ?
A statement is a true or false. Statements?
3 = 4 – 1 ? Statement!
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Statements?
  \(3 = 4 - 1\) ? Statement!
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  \(3\) ?
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- $3 = 4 - 1$ ? Statement!
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Statements?

3 = 4 − 1 ? Statement!
3 = 5 ? Statement!
3 ? Not a statement!

n = 3 ?
First there was logic...

A statement is a true or false.
Statements?
  $3 = 4 - 1$ ? Statement!
  $3 = 5$ ? Statement!
  $3$ ? Not a statement!
  $n = 3$ ? Not a statement...
First there was logic...

A statement is a true or false.

Statements?

\[ 3 = 4 - 1 \] ? Statement!
\[ 3 = 5 \] ? Statement!
\[ 3 \] ? Not a statement!
\[ n = 3 \] ? Not a statement...but a predicate.
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Statements?
  \(3 = 4 - 1\) ? Statement!
  \(3 = 5\) ? Statement!
  \(3\) ? Not a statement!
  \(n = 3\) ? Not a statement...but a predicate.

**Predicate:** Statement with free variable(s).
First there was logic...

A statement is a true or false.

Statements?

3 = 4 − 1 ? Statement!
3 = 5 ? Statement!
3 ? Not a statement!

n = 3 ? Not a statement...but a predicate.

Predicate: Statement with free variable(s).

Example: \( x = 3 \)
First there was logic...

A statement is a true or false.
Statements?
- $3 = 4 - 1$ ? Statement!
- $3 = 5$ ? Statement!
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**Predicate:** Statement with free variable(s).
- Example: $x = 3$  Given a value for $x$, becomes a statement.
A statement is a true or false.

Statements?

3 = 4 − 1 ? Statement!
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**Predicate:** Statement with free variable(s).

Example: $x = 3$  Given a value for $x$, becomes a statement.

Predicate?

$n > 3$ ?
First there was logic...

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Statements?

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Predicate: Statement with free variable(s).

Example: $x = 3$   Given a value for $x$, becomes a statement.

Predicate?

$n > 3$ ? Predicate: $P(n)$!
First there was logic...

A statement is a true or false.
Statements?

\[ 3 = 4 - 1 \ ? \text{ Statement!} \]
\[ 3 = 5 \ ? \text{ Statement!} \]
\[ 3 \ ? \text{ Not a statement!} \]
\[ n = 3 \ ? \text{ Not a statement...but a predicate.} \]

**Predicate:** Statement with free variable(s).

Example: \( x = 3 \)  
Given a value for \( x \), becomes a statement.

Predicate?

\[ n > 3 \ ? \text{ Predicate: } P(n)! \]
\[ x = y? \]
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  \[ 3 = 4 - 1 \] ? Statement!
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**Predicate:** Statement with free variable(s).
Example: \( x = 3 \)  Given a value for \( x \), becomes a statement.
Predicate?
  \( n > 3 \) ? Predicate: \( P(n) \)!
  \( x = y \) ? Predicate: \( P(x, y) \)!
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  \(3 = 4 - 1\) ? Statement!
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**Predicate:** Statement with free variable(s).

Example: \(x = 3\) Given a value for \(x\), becomes a statement.

Predicate?
  \(n > 3\) ? Predicate: \(P(n)\)!
  \(x = y\)? Predicate: \(P(x, y)\)!
  \(x + y\)?
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**Predicate: Statement with free variable(s).**

Example: \( x = 3 \)  Given a value for \( x \), becomes a statement.

Predicate?

\( n > 3 \) ? Predicate: \( P(n) \)!
\( x = y \) ? Predicate: \( P(x, y) \)!
\( x + y \) ? No.
First there was logic...

**A statement is a true or false.**

Statements?

3 = 4 − 1 ? Statement!
3 = 5 ? Statement!
3 ? Not a statement!

\( n = 3 \) ? Not a statement...but a **predicate**.

**Predicate:** **Statement with free variable(s).**

Example: \( x = 3 \)  
Given a value for \( x \), becomes a statement.

Predicate?

\( n > 3 \) ? Predicate: \( P(n)! \)

\( x = y \)? Predicate: \( P(x,y)! \)

\( x + y \)? No. An expression, not a statement.
A statement is a true or false. Statements?
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**Predicate: Statement with free variable(s).**
Example: $x = 3$ Given a value for $x$, becomes a statement.

Predicate?
- $n > 3$? Predicate: $P(n)$!
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**Quantifiers:**
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**Quantifiers:**
\( (\forall x) \ P(x) \).
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**Quantifiers:**

- $(\forall x) P(x)$. For every $x$, $P(x)$ is true.
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**Quantifiers:**
- $(\forall x) P(x)$. For every $x$, $P(x)$ is true.
- $(\exists x) P(x)$. 
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n > 3 ? Predicate: \( P(n) \)!
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Quantifiers:
\((\forall x) \ P(x)\).  For every \( x \), \( P(x) \) is true.
\((\exists x) \ P(x)\).  There exists an \( x \), where \( P(x) \) is true.


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  $(\forall n \in N), n^2 \geq n$.  
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$(\forall x \in R)(\exists y \in R)y > x$. 

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Connecting Statements

\[ A \land B, A \lor B, \neg A. \]
Connecting Statements

\[ A \wedge B, \ A \vee B, \ \neg A. \]

You got this!
Connecting Statements

\[ A \land B, \ A \lor B, \ \neg A. \]

You got this!

Propositional Expressions and Logical Equivalence
Connecting Statements

\[ A \land B, \ A \lor B, \ \neg A. \]

You got this!

Propositional Expressions and Logical Equivalence

\[(A \implies B) \equiv (\neg A \lor B)\]
Connecting Statements

\( A \land B, A \lor B, \neg A. \)

You got this!

Propositional Expressions and Logical Equivalence

\[ (A \implies B) \equiv (\neg A \lor B) \]
\[ \neg(A \lor B) \equiv (\neg A \land \neg B) \]
Connecting Statements

\( A \land B, A \lor B, \neg A. \)

You got this!

Propositional Expressions and Logical Equivalence

\[
(A \implies B) \equiv (\neg A \lor B) \\
\neg (A \lor B) \equiv (\neg A \land \neg B)
\]
You got this!

Propositional Expressions and Logical Equivalence

\[(A \implies B) \equiv (\neg A \lor B)\]
\[\neg (A \lor B) \equiv (\neg A \land \neg B)\]

Proofs: truth table or manipulation of known formulas.
Connecting Statements

\(A \land B, A \lor B, \neg A.\)

You got this!

Propositional Expressions and Logical Equivalence

\[(A \implies B) \equiv (\neg A \lor B)\]
\[\neg (A \lor B) \equiv (\neg A \land \neg B)\]

Proofs: truth table or manipulation of known formulas.

\[(\forall x)(P(x) \land Q(x)) \equiv (\forall x)P(x) \land (\forall x)Q(x)\]
..and then proofs...

Direct: $P \implies Q$

Example: $a$ is even $\implies a^2$ is even.

Approach: What is even?

$a = 2k$

$a^2 = 4k^2$.

What is even?

$a^2 = 2(2k^2)$.

Integers closed under multiplication!

$a^2$ is even.

Contrapositive:

$P \implies Q$ or $\neg Q \implies \neg P$.

Example: $a^2$ is odd $\implies a$ is odd.

Contrapositive: $a$ is even $\implies a^2$ is even.

Contradiction:

$P \neg P \implies \text{false}$

$\neg P \implies R \land \neg R$.

Useful for prove something does not exist:

Example: rational representation of $\sqrt{2}$ does not exist.

Example: finite set of primes does not exist.

Example: rogue couple does not exist.
..and then proofs...

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$a^2 = 2(2k^2)$
..and then proofs...

Direct: $P \implies Q$

Example: $a$ is even implies $a^2$ is even.

Approach: What is even? $a = 2k$

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What is even?

$a^2 = 2(2k^2)$

Integers closed under multiplication!

Contrapositive: $P \implies Q$ or $\neg Q \implies \neg P$.

Example: $a^2$ is odd implies $a$ is odd.

Contrapositive: $a$ is even implies $a^2$ is even.

Contradiction: $P \implies \neg P$.

$\neg P \implies R \land \neg R$.

Useful for prove something does not exist:

Example: rational representation of $\sqrt{2}$ does not exist.

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Example: \( a \) is even \( \implies \) \( a^2 \) is even.

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**Contrapositive:** \( P \implies Q \) or \( \neg Q \implies \neg P \).
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Contrapositive: $P \implies Q$ or $\neg Q \implies \neg P$.

Example: $a^2$ is odd $\implies a$ is odd.
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**Example:** \( a^2 \) is odd \( \implies a \) is odd.

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\( \neg P \implies \textbf{false} \)
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Example: $a^2$ is odd $\implies a$ is odd.
Contrapositive: $a$ is even $\implies a^2$ is even.

Contradiction: $P$
$\neg P \implies \text{false}$
$\neg P \implies R \land \neg R$
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Useful for prove something does not exist:
..and then proofs...

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Useful for prove something does not exist:
Example: rational representation of $\sqrt{2}$
..and then proofs...

Direct: \( P \implies Q \)

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Example: \( a^2 \) is odd \( \implies a \) is odd.

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Contradiction: \( P \)
\[
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Useful for prove something does not exist:
Example: rational representation of \( \sqrt{2} \) does not exist.
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Direct: $P \implies Q$

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Example: rational representation of $\sqrt{2}$ does not exist.
Example: finite set of primes
..and then proofs...

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Example: \( a^2 \) is odd \( \implies \) \( a \) is odd.
Contrapositive: \( a \) is even \( \implies \) \( a^2 \) is even.

Contradiction: \( P \)
\[ \neg P \implies \text{false} \]
\[ \neg P \implies R \land \neg R \]

Useful for prove something does not exist:
Example: rational representation of \( \sqrt{2} \) does not exist.
Example: finite set of primes does not exist.
Example: rogue couple does not exist.
...jumping forward..

Contradiction in induction:
...jumping forward..

Contradiction in induction:
contradict place where induction step doesn’t hold.
...jumping forward..

Contradiction in induction:
contradict place where induction step doesn’t hold.

Well Ordering Principle.
...jumping forward..

Contradiction in induction:
contradict place where induction step doesn’t hold.

Well Ordering Principle.
Stable Marriage:
...jumping forward..

Contradiction in induction:
contradict place where induction step doesn’t hold.

Well Ordering Principle.
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first day where women does not improve.
...jumping forward..

Contradiction in induction:
  contradict place where induction step doesn’t hold.

Well Ordering Principle.
Stable Marriage:
  first day where women does not improve.
  first day where any man rejected by optimal women.
Contradiction in induction:
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Do not exist.
...jumping forward..

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Well Ordering Principle.
  Stable Marriage:
    first day where women does not improve.
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  Do not exist.
...and then induction...

\[ P(0) \land ((\forall n)(P(n) \implies P(n + 1)) \equiv (\forall n \in \mathbb{N}) P(n). \]
...and then induction...

\[ P(0) \land ((\forall n)(P(n) \implies P(n+1)) \equiv (\forall n \in N) P(n)). \]

**Thm:** For all \( n \geq 1 \), \( 8 \mid 3^{2n} - 1 \).
...and then induction...

\[ P(0) \land (\forall n)(P(n) \implies P(n + 1)) \equiv (\forall n \in \mathbb{N}) P(n). \]

**Thm:** For all \( n \geq 1, 8 \mid 3^{2n} - 1. \)

Induction on \( n. \)
...and then induction...

\[ P(0) \land ((\forall n)(P(n) \implies P(n+1)) \equiv (\forall n \in \mathbb{N}) P(n)). \]

**Thm:** For all \( n \geq 1, 8 \mid 3^{2n} - 1. \)

Induction on \( n. \)

Base: \( 8 \mid 3^2 - 1. \)
...and then induction...

\[ P(0) \land ((\forall n)(P(n) \implies P(n+1)) \equiv (\forall n \in \mathbb{N}) P(n)). \]

**Thm:** For all \( n \geq 1 \), \( 8|3^{2n} - 1 \).

Induction on \( n \).

Base: \( 8|3^2 - 1 \).
...and then induction...

\[ P(0) \land (\forall n)(P(n) \implies P(n+1)) \equiv (\forall n \in \mathbb{N}) P(n). \]

**Thm:** For all \( n \geq 1, \) \( 8 \mid 3^{2n} - 1. \)

Induction on \( n. \)

Base: \( 8 \mid 3^2 - 1. \)

Induction Hypothesis: Assume \( P(n): \) True for some \( n. \)
...and then induction...

\[ P(0) \land ((\forall n)(P(n) \implies P(n+1)) \equiv (\forall n \in \mathbb{N}) P(n)). \]

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Induction on \( n \).

Base: \( 8 | 3^2 - 1 \).

Induction Hypothesis: Assume \( P(n) \): True for some \( n \).

Induction Step: Prove \( P(n+1) \)
...and then induction...

\[ P(0) \land ((\forall n)(P(n) \implies P(n+1)) \equiv (\forall n \in N) P(n). \]

**Thm:** For all \( n \geq 1 \), \( 8 | 3^{2n} - 1 \).

Induction on \( n \).

Base: \( 8 | 3^2 - 1 \).

Induction Hypothesis: Assume \( P(n) \): True for some \( n \).

Induction Step: Prove \( P(n+1) \)

\[ 3^{2n+2} - 1 = \]
...and then induction...

\[ P(0) \land ((\forall n)(P(n) \Rightarrow P(n+1)) \equiv (\forall n \in \mathbb{N}) P(n). \]

**Thm:** For all \( n \geq 1, 8 \mid 3^{2n} - 1. \)

Induction on \( n. \)

Base: \( 8 \mid 3^2 - 1. \)

Induction Hypothesis: Assume \( P(n): \) True for some \( n. \)

Induction Step: Prove \( P(n+1) \)
\[ 3^{2(n+2)} - 1 = 9(3^{2n}) - 1 \]
...and then induction...

\[ P(0) \land ((\forall n) (P(n) \implies P(n+1)) \equiv (\forall n \in \mathbb{N}) P(n). \]

**Thm:** For all \( n \geq 1 \), \( 8 \mid 3^{2n} - 1 \).

Induction on \( n \).

Base: \( 8 \mid 3^2 - 1 \).

Induction Hypothesis: Assume \( P(n) \): True for some \( n \).

\[(3^{2n} - 1 = 8d)\]

Induction Step: Prove \( P(n+1) \)

\[3^{2n+2} - 1 = 9(3^{2n}) - 1 \quad (\text{by induction hypothesis})\]
...and then induction...

\[ P(0) \land ((\forall n)(P(n) \implies P(n+1)) \equiv (\forall n \in \mathbb{N}) P(n). \]

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Induction on \( n \).

Base: \( 8 \mid 3^2 - 1 \).

Induction Hypothesis: Assume \( P(n) \): True for some \( n \).

\( 3^{2n} - 1 = 8d \)

Induction Step: Prove \( P(n + 1) \)

\[ 3^{2n+2} - 1 = 9(3^{2n}) - 1 \quad \text{(by induction hypothesis)} \]
\[ = 9(8d + 1) - 1 \]
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Induction Hypothesis: Assume \( P(n) \): True for some \( n \).
\[ (3^{2n} - 1 = 8d) \]

Induction Step: Prove \( P(n + 1) \)
\[ 3^{2n+2} - 1 = 9(3^{2n}) - 1 \] (by induction hypothesis)
\[ = 9(8d + 1) - 1 \]
\[ = 72d + 8 \]
...and then induction...

\[ P(0) \land ((\forall n)(P(n) \implies P(n+1)) \equiv (\forall n \in \mathbb{N}) P(n)). \]

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Induction on \( n \).

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\( 3^{2n} - 1 = 8d \) \( \quad (3^{2n} - 1 = 8d) \)

Induction Step: Prove \( P(n+1) \)

\[ 3^{2n+2} - 1 = 9(3^{2n}) - 1 \quad \text{(by induction hypothesis)} \]
\[ = 9(8d + 1) - 1 \]
\[ = 72d + 8 \]
\[ = 8(9d + 1) \]
...and then induction...

\[ P(0) \land ((\forall n)(P(n) \implies P(n+1)) \equiv (\forall n \in \mathbb{N}) P(n)). \]

**Thm:** For all \( n \geq 1, 8 | 3^{2n} - 1. \)

Induction on \( n. \)

Base: \( 8 | 3^2 - 1. \)

Induction Hypothesis: Assume \( P(n): \) True for some \( n. \)

\( (3^{2n} - 1 = 8d) \)

Induction Step: Prove \( P(n+1) \)

\[
3^{2n+2} - 1 = 9(3^{2n}) - 1 \quad \text{(by induction hypothesis)} \\
= 9(8d + 1) - 1 \\
= 72d + 8 \\
= 8(9d + 1)
\]

Divisible by 8.
...and then induction...

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Induction Hypothesis: Assume \( P(n) \): True for some \( n \).
\[ (3^{2n} - 1 = 8d) \]

Induction Step: Prove \( P(n+1) \)
\[
3^{2n+2} - 1 = 9(3^{2n}) - 1 \quad \text{(by induction hypothesis)}
\]
\[
= 9(8d + 1) - 1
\]
\[
= 72d + 8
\]
\[
= 8(9d + 1)
\]

Divisible by 8.
Stable Marriage: a study in definitions and WOP.

$n$-men, $n$-women.
Stable Marriage: a study in definitions and WOP.

$n$-men, $n$-women.
Each person has completely ordered preference list
Stable Marriage: a study in definitions and WOP.

$n$-men, $n$-women.

Each person has completely ordered preference list contains every person of opposite gender.
Stable Marriage: a study in definitions and WOP.

$n$-men, $n$-women.

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Pairing.
Stable Marriage: a study in definitions and WOP.

$n$-men, $n$-women.

Each person has completely ordered preference list contains every person of opposite gender.

Pairing.
Set of pairs $(m_i, w_j)$ containing all people *exactly* once.
Stable Marriage: a study in definitions and WOP.

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**Pairing.**
Set of pairs $(m_i, w_j)$ containing all people *exactly* once.
How many pairs?
Stable Marriage: a study in definitions and WOP.

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**Pairing.**
Set of pairs $(m_i, w_j)$ containing all people *exactly* once.

How many pairs? $n$. 
Stable Marriage: a study in definitions and WOP.

$n$-men, $n$-women.

Each person has completely ordered preference list contains every person of opposite gender.

**Pairing.**
Set of pairs $(m_i, w_j)$ containing all people *exactly* once.

- How many pairs? $n$.
- People in pair are **partners** in pairing.

Rogue Couple in a pairing.
A $m_j$ and $w_k$ who like each other more than their partners.

Stable Pairing.
Pairing with no rogue couples.

Does stable pairing exist?
No, for roommates problem.
Stable Marriage: a study in definitions and WOP.

$n$-men, $n$-women.

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**Pairing.**
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How many pairs? $n$.
People in pair are *partners* in pairing.

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Does stable pairing exist?
No, for roommates problem.
Traditional Marriage Algorithm:

Each Day:
- All men propose to favorite woman who has not yet rejected him.
- Every woman rejects all but best men who proposes.

Useful Algorithmic Definitions:
- Man crosses off woman who rejected him.
- Woman's current proposer is "on string."

"Propose and Reject."
- Either men propose or women. But not both.
- Traditional propose and reject where men propose.

Key Property: Improvement Lemma:
- Every day, if man on string for woman, \( \Rightarrow \) any future man on string is better.

Stability:
- No rogue couple.
  - Rogue couple \((M, W)\) \( \Rightarrow \) M proposed to W \( \Rightarrow \) W ended up with someone she liked better than M.

Not rogue couple!
TMA.

Traditional Marriage Algorithm:

Each Day:
TMA.

Traditional Marriage Algorithm:

Each Day:
All men propose to favorite woman who has not yet rejected him.
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  All men propose to favorite woman who has not yet rejected him.
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Stability:
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Every day, if man on string for woman,
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Stability: No rogue couple.
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rogue couple (M,W)
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- rogue couple (M,W)
  \[ \rightarrow M \text{ proposed to W} \]
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rogue couple \((M,W)\)
\[ \implies M \text{ proposed to } W \]
\[ \implies W \text{ ended up with someone she liked better than } M. \]
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rogue couple (M,W)
M proposed to W
W ended up with someone she liked better than M.
Not rogue couple!
Optimality/Pessimal

Optimal partner if best partner in any stable pairing.
Optimality/Pessimal

Optimal partner if best partner in any stable pairing. Not necessarily first in list.
Optimality/Pessimal

Optimal partner if best partner in any stable pairing.
Not necessarily first in list.
Possibly no stable pairing with that partner.
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Optimal partner if best partner in any stable pairing.
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Man-optimal pairing is pairing where every man gets optimal partner.

Thm: \( TMA \) produces male optimal pairing, \( S \).
First man \( M \) to lose optimal partner.
Better partner \( W \) for \( M \).
Different stable pairing \( T \).

\( TMA: \) \( M \) asked \( W \) first!
There is \( M' \) who bumps \( M \) in \( TMA \).
\( W \) prefers \( M' \).
\( M' \) likes \( W \) at least as much as optimal partner.
Not first bump.
\( M' \) and \( W \) is rogue couple in \( T \).

Thm: Woman pessimal.
Man optimal \( \Rightarrow \) Woman pessimal.
Woman optimal \( \Rightarrow \) Man pessimal.
Optimality/Pessimal

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**Thm:** Woman pessimal.
Man optimal $\Rightarrow$ Woman pessimal.
Woman optimal $\Rightarrow$ Man pessimal.
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There is $M'$ who bumps $M$ in TMA.
$W$ prefers $M'$. 
Optimality/Pessimal

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Possibly no stable pairing with that partner.

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There is $M'$ who bumps $M$ in TMA.
$W$ prefers $M'$.
$M'$ likes $W$ at least as much as optimal partner.
Optimality/Pessimal

Optimal partner if best partner in any stable pairing.
Not necessarily first in list.
Possibly no stable pairing with that partner.

Man-optimal pairing is pairing where every man gets optimal partner.

**Thm:** TMA produces male optimal pairing, S.
First man $M$ to lose optimal partner.
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Graphs

\[ G = (V, E) \]

- Set of vertices.
- Set of edges.
- Directed: ordered pair of vertices.
- Adjacent, Incident, Degree.
- In-degree, Out-degree.

Thm: Sum of degrees is twice the number of edges |

Degree of vertices is total incidences.

Pair of Vertices are Connected: If there is a path between them.

Connected Component: maximal set of connected vertices.

Connected Graph: one connected component.
Graphs

\[ G = (V, E) \]
\[ V \text{ - set of vertices.} \]
$G = (V, E)$
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Graph Algorithm: Eulerian Tour

**Thm:** Every connected graph where every vertex has even degree has an Eulerian Tour; a tour which visits every edge exactly once.
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- Take a walk using each edge at most once.

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**Property:**
- Return to starting point.

**Proof Idea:** Even degree.
- Recurse on connected components.
- Put together.

**Property:**
- Walk visits every component.

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Graph Coloring.

Given $G = (V, E)$, a coloring of a $G$ assigns colors to vertices $V$ where for each edge the endpoints have different colors.
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Notice that the last one, has one three colors.
Given $G = (V, E)$, a coloring of a $G$ assigns colors to vertices $V$ where for each edge the endpoints have different colors.

Notice that the last one, has one three colors. Fewer colors than number of vertices.
Graph Coloring.

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Notice that the last one, has one three colors.
Fewer colors than number of vertices.
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Interesting things to do.
Graph Coloring.

Given $G = (V, E)$, a coloring of a $G$ assigns colors to vertices $V$ where for each edge the endpoints have different colors.

Notice that the last one, has one three colors.
Fewer colors than number of vertices.
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Interesting things to do. Algorithm!
Planar graphs and maps.

Planar graph coloring $\equiv$ map coloring.
Planar graphs and maps.

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Four color theorem is about planar graphs!
Theorem: Every planar graph can be colored with six colors.
Six color theorem.

**Theorem:** Every planar graph can be colored with six colors.

**Proof:**
Six color theorem.

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**Proof:**
Recall: \( e \leq 3v - 6 \) for any planar graph.
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Recall: $e \leq 3v - 6$ for any planar graph.
From Euler’s Formula.
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**Theorem:** Every planar graph can be colored with six colors.

**Proof:**
Recall: $e \leq 3v - 6$ for any planar graph.
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Total degree: $2e$
Theorem: Every planar graph can be colored with six colors.

Proof:
Recall: $e \leq 3v - 6$ for any planar graph.
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Total degree: $2e$
Average degree: $\leq \frac{2e}{v}$
Six color theorem.

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Recall: $e \leq 3v - 6$ for any planar graph.
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Total degree: \( 2e \)
Average degree: \( \leq \frac{2e}{v} \leq \frac{2(3v-6)}{v} \leq 6 - \frac{12}{v} \).
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There exists a vertex with degree $< 6$
**Theorem:** Every planar graph can be colored with six colors.

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There exists a vertex with degree < 6 or at most 5.
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There exists a vertex with degree $< 6$ or at most 5.
   Remove vertex $v$ of degree at most 5.
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   Inductively color remaining graph.
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Color is available for $v$ since only five neighbors...
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Five color theorem

Theorem: Every planar graph can be colored with five colors.
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Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.
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Proof:
Again with the degree 5 vertex.
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Proof:
Again with the degree 5 vertex. Again recurse.
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Assume neighbors are colored all differently.

\[ \text{Planar.} \quad \Rightarrow \quad \text{paths intersect at a vertex!} \]

What color is it?
Must be blue or green to be on that path.
Must be red or orange to be on that path.
Contradiction.
Can recolor one of the neighbors.
And recolor “center” vertex.
Theorem: Every planar graph can be colored with five colors.

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Again with the degree 5 vertex. Again recurse.

Assume neighbors are colored all differently. Otherwise done.
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Proof:
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Assume neighbors are colored all differently. Otherwise done.
Switch green to blue in component.

Planar. =⇒ paths intersect at a vertex!
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![Diagram of a planar graph with vertices colored with five different colors.](image-url)
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Assume neighbors are colored all differently. Otherwise done.
Switch green to blue in component.
Done. Unless blue-green path to blue.
Switch red to orange in its component.
Done. Unless red-orange path to red.

Planar. \(\Rightarrow\) paths intersect at a vertex!
What color is it?
Must be blue or green to be on that path.
Five color theorem

Theorem: Every planar graph can be colored with five colors.

Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.

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Four Color Theorem
Four Color Theorem

**Theorem:** Any planar graph can be colored with four colors.
Four Color Theorem

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**Proof:**
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Proof: Not Today!
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Proof: Not Today!
Graph Types: Complete Graph.

- $K_n$, $|V| = n$; every edge present.
- Degree of vertex: $|V| - 1$.
- Very connected; lots of edges: $n(n-1)/2$. 

![Graph Types: Complete Graph.](image)
Graph Types: Complete Graph.

\[ K_n, |V| = n \]
Graph Types: Complete Graph.

$K_n, \ |V| = n$

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Graph Types: Complete Graph.

$k_n$, $|V| = n$

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Trees.

Definitions:
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A connected graph without a cycle.
Trees.

Definitions:

A connected graph without a cycle.
A connected graph with $|V| - 1$ edges.
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Definitions:

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- A connected graph with $|V| - 1$ edges.
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To tree or not to tree!
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Minimally connected, minimum number of edges to connect.
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To tree or not to tree!

- Minimally connected, minimum number of edges to connect.

Property:
- Can remove a single node and break into components of size at most $|V|/2$. 
Hypercubes.

Also represents bit-strings nicely.

\[ G = (V, E) \]

\[ V = \{0, 1\}^n \]

\[ E = \{ (x, y) | x \text{ and } y \text{ differ in one bit position.} \} \]
Hypercubes. Really connected.
Hypercube

Hypercubes. Really connected. $|V| \log |V|$ edges!
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Recursive Definition.

A 0-dimensional hypercube is a node labelled with the empty string of bits.
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An \( n \)-dimensional hypercube consists of a 0-subcube (1-subcube) which is a \( n - 1 \)-dimensional hypercube with nodes labelled \( 0x \) \((1x)\) with the additional edges \((0x, 1x)\).
Recursive Definition.

A 0-dimensional hypercube is a node labelled with the empty string of bits.

An $n$-dimensional hypercube consists of a 0-subcube (1-subcube) which is a $n-1$-dimensional hypercube with nodes labelled $0x$ ($1x$) with the additional edges $(0x, 1x)$. 
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![Diagram of an n-dimensional hypercube]
Hypercube: properties

Rudrata Cycle: cycle that visits every node.
Rudrata Cycle: cycle that visits every node. Eulerian?

FYI: Also cuts represent boolean functions.

Nice Paths between nodes.

Get from 000100 to 101000.

000100 → 100100 → 101100 → 101000

Correct bits in string, moves along path in hypercube!

Good communication network!
Rudrata Cycle: cycle that visits every node. Eulerian? If $n$ is even.
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Good communication network!
Arithmetic modulo $m$.
Elements of equivalence classes of integers.
Modular Arithmetic...

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$\{0, \ldots, m - 1\}$
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if $i = a + km$ for integer $k$. 

Negative numbers work the way you are used to.
$-3 \equiv 0 \pmod{7}$
$-3 \equiv 7 \pmod{7}$
$-3 \equiv 4 \pmod{7}$ 

Additive inverses are intuitively negative numbers.
Arithmetic modulo $m$. Elements of equivalence classes of integers. 
\{0, \ldots, m-1\}
and integer $i \equiv a \pmod{m}$
if $i = a + km$ for integer $k$.
or if the remainder of $i$ divided by $m$ is $a$.

Can do calculations by taking remainders at the beginning, in the middle or at the end.

$58 + 32 = 90 = 6 \pmod{7}$

$58 + 32 = 2 + 4 = 6 \pmod{7}$

$58 + 32 = 2 + -3 = -1 = 6 \pmod{7}$

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...Modular Arithmetic...

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Additive inverses are intuitively negative numbers.
Modular Arithmetic and multiplicative inverses.

\[ 3^{-1} \pmod{7} \]?

Inverse Unique?
Yes.
Proof:
a and b inverses of x (mod n)

\[ ax = bx = 1 \pmod{n} \]

\[ axb = bxb = b \pmod{n} \]

\[ a = b \pmod{n} \].

\[ 3^{-1} \pmod{6} \]?

No, no, no....

\{ 3(1), 3(2), 3(3), 3(4), 3(5) \}

\{ 3, 6, 3, 6, 3 \}

See, ... no inverse!
Modular Arithmetic and multiplicative inverses.

\[ 3^{-1} \pmod{7} \equiv 5 \]
Modular Arithmetic and multiplicative inverses.

\[ 3^{-1} \pmod{7} \, ? \, 5 \]
\[ 5^{-1} \pmod{7} \, ? \]
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Proof: \(a\) and \(b\) inverses of \(x \pmod{n}\)

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{$3(1), 3(2), 3(3), 3(4), 3(5)$}
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\]
\[
\{3, 6, 3, 6, 3\}
Modular Arithmetic and multiplicative inverses.

$3^{-1} \pmod{7}$? 5  
$5^{-1} \pmod{7}$? 3

Inverse Unique? Yes.

Proof: $a$ and $b$ inverses of $x \pmod{n}$

\begin{align*}
ax &= bx = 1 \pmod{n} \\
axb &= bxb = b \pmod{n} \\
a &= b \pmod{n}.
\end{align*}

$3^{-1} \pmod{6}$? No, no, no....

\begin{align*}
\{3(1), 3(2), 3(3), 3(4), 3(5)\} \\
\{3, 6, 3, 6, 3\}
\end{align*}

See,
Modular Arithmetic and multiplicative inverses.

\[ 3^{-1} \pmod{7} \text{? } 5 \]
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Inverse Unique? Yes.

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\[ ax = bx = 1 \pmod{n} \]
\[ axb = bxb = b \pmod{n} \]
\[ a = b \pmod{n}. \]

\[ 3^{-1} \pmod{6} \text{? } \text{No, no, no....} \]

\[ \{3(1), 3(2), 3(3), 3(4), 3(5)\} \]
\[ \{3, 6, 3, 6, 3\} \]

See,... no inverse!
Modular Arithmetic Inverses and GCD

$x$ has inverse modulo $m$ if and only if $\gcd(x, m) = 1$. 

Finding $\gcd$.

\[
\gcd(x, y) = \gcd(y, x - y) = \gcd(y, x \mod y)
\]

Give recursive algorithm!

Base Case?

\[
\gcd(x, 0) = x
\]

Extended-gcd($x, y$) returns $(d, a, b)$ such that $d = \gcd(x, y)$ and $d = ax + by$.

Idea: egcd.

\[\gcd\text{ produces 1 by adding and subtracting multiples of } x \text{ and } y\]
Modular Arithmetic Inverses and GCD

$x$ has inverse modulo $m$ if and only if $gcd(x, m) = 1$. Group structures more generally.
Modular Arithmetic Inverses and GCD

$x$ has inverse modulo $m$ if and only if $gcd(x, m) = 1$.

Group structures more generally.

Proof Idea:

$\{0 \cdot x, \ldots, (m - 1) \cdot x\}$ are distinct modulo $m$ if and only if $gcd(x, m) = 1$. 

Finding gcd.

$gcd(x, y) = gcd(y, x - y)$.

Give recursive Algorithm!

Base Case?

$gcd(x, 0) = x$.

Extended-gcd($x, y$) returns $(d, a, b)$ $d = gcd(x, y)$ and $d = ax + by$.

Multiplicative inverse of $(x, m)$.

egcd($x, m$) = $(1, a, b)$ $a$ is inverse!

$1 = ax + bm = ax \pmod{m}$.
Modular Arithmetic Inverses and GCD

$x$ has inverse modulo $m$ if and only if $gcd(x, m) = 1$.

Group structures more generally.

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\gcd(x, y) = \gcd(y, x - y) = \gcd(y, x \mod y).
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Modular Arithmetic Inverses and GCD

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Group structures more generally.

Proof Idea:
{0, x, ..., (m - 1)x} are distinct modulo $m$ if and only if $gcd(x, m) = 1$.

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Extended-gcd($x, y$)
Modular Arithmetic Inverses and GCD

$x$ has inverse modulo $m$ if and only if $gcd(x, m) = 1$.

Group structures more generally.

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Extended-$gcd(x, y)$ returns $(d, a, b)$
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\[ d = gcd(x, y) \]
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Multiplicative inverse of $(x, m)$. 

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Finding gcd.
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\[egcd(x, m) = (1, a, b)\]
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\[ egcd(x, m) = (1, a, b) \]
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Modular Arithmetic Inverses and GCD

\(x\) has inverse modulo \(m\) if and only if \(\text{gcd}(x, m) = 1\).

Group structures more generally.

Proof Idea:
\(\{0x, \ldots, (m-1)x\}\) are distinct modulo \(m\) if and only if \(\text{gcd}(x, m) = 1\).

Finding gcd.
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Give recursive Algorithm! Base Case? \(\text{gcd}(x, 0) = x\).

Extended-gcd\((x, y)\) returns \((d, a, b)\)
\[
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\]

Multiplicative inverse of \((x, m)\).
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Idea: egcd.
Modular Arithmetic Inverses and GCD

$x$ has inverse modulo $m$ if and only if $gcd(x, m) = 1$.

Group structures more generally.

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Multiplicative inverse of $(x, m)$.

$egcd(x, m) = (1, a, b)$

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Idea: egcd.

gcd produces 1
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Idea: egcd.
\[ gcd \text{ produces 1} \]
\[ \text{by adding and subtracting multiples of } x \text{ and } y \]
Modular Arithmetic Inverses and GCD

$x$ has inverse modulo $m$ if and only if $gcd(x, m) = 1$.

Group structures more generally.

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Idea: egcd.
\( gcd \) produces 1
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Example: $p = 7$, $q = 11$. 
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$N = 77$. 
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$(p - 1)(q - 1) = 60$
Example: $p = 7, \ q = 11$.

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$(p - 1)(q - 1) = 60$

Choose $e = 7$, since $\gcd(7, 60) = 1$. 

egcd(7, 60).

$7(0) + 60(1) = 60$

$7(1) + 60(0) = 7$

$7(−8) + 60(1) = 4$

$7(9) + 60(−1) = 3$

$7(−17) + 60(2) = 1$

Confirm: $−119 + 120 = 1$

$d = e - 1 = −17 = 43 \pmod{60}$
Example: $p = 7$, $q = 11$.

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$(p - 1)(q - 1) = 60$

Choose $e = 7$, since $\gcd(7, 60) = 1$.

$\text{egcd}(7, 60)$. 

$7(0) + 60(1) = 60$

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$d = e - 1 = 43 = (\mod 60)$
Example: $p = 7$, $q = 11$.

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Example: $p = 7, \ q = 11$.

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Choose $e = 7$, since $\gcd(7, 60) = 1$.

$\text{egcd}(7, 60)$.

\[
egin{align*}
7(0) + 60(1) &= 60 \\
7(1) + 60(0) &= 7
\end{align*}
\]
Example: $p = 7, \ q = 11$.

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Choose $e = 7$, since $\gcd(7, 60) = 1$.

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\[
\begin{align*}
7(0) + 60(1) &= 60 \\
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\end{align*}
\]
Example: \( p = 7, \ q = 11. \)

\[ N = 77. \]
\[ (p - 1)(q - 1) = 60 \]

Choose \( e = 7, \) since \( \gcd(7, 60) = 1. \)

\( \gcd(7, 60). \)

\[
\begin{align*}
7(0) + 60(1) & = 60 \\
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Confirm:
Example: \(p = 7, \ q = 11\).

\[N = 77.\]

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\[\text{egcd}(7, 60).\]

\[
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Confirm: \(-119 + 120 = 1\)
Example: $p = 7, \ q = 11$.

$N = 77$.

$(p - 1)(q - 1) = 60$

Choose $e = 7$, since $\text{gcd}(7,60) = 1$.

\[
\text{egcd}(7,60).
\]

\[
\begin{align*}
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\end{align*}
\]

Confirm: $-119 + 120 = 1$

$d = e^{-1} = -17 = 43 = (\text{mod}\ 60)$
Fermat from Bijection.

**Fermat’s Little Theorem:** For prime $p$, and $a \not\equiv 0 \pmod{p}$,
Fermat from Bijection.

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$$a^{p-1} \equiv 1 \pmod{p}.$$
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**Fermat’s Little Theorem:** For prime $p$, and $a \not\equiv 0 \pmod{p}$,

\[ a^{p-1} \equiv 1 \pmod{p}. \]

**Proof:** Consider $T = \{ a \cdot 1 \pmod{p}, \ldots, a \cdot (p-1) \pmod{p} \}$.
Fermat from Bijection.

**Fermat’s Little Theorem:** For prime $p$, and $a \not\equiv 0 \pmod{p}$,

$$a^{p-1} \equiv 1 \pmod{p}.$$  

**Proof:** Consider $T = \{a \cdot 1 \pmod{p}, \ldots, a \cdot (p - 1) \pmod{p}\}$.

$T$ is range of function $f(x) = ax \pmod{p}$ for set $S = \{1, \ldots, p - 1\}$.  

Since multiplication is commutative,

$$a^{p-1} \equiv 1 \pmod{p}.$$
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Invertible function:
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Invertible function: one-to-one.
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$T$ is range of function $f(x) = ax \pmod{p}$ for set $S = \{1, \ldots, p-1\}$. 

Invertible function: one-to-one. 

$T \subseteq S$ since $0 \not\in T$. 

Since multiplication is commutative. 

$a \cdot (p-1) \cdot (1 \cdot \ldots \cdot (p-1)) \equiv (1 \cdot \ldots \cdot (p-1)) \pmod{p}$. 

Each of $2, \ldots, (p-1)$ has an inverse modulo $p$, multiply by inverses to get... 

$a \cdot (p-1) \equiv 1 \pmod{p}$. 

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$$a^{p-1} \equiv 1 \pmod{p}.$$  

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$$\implies T = S.$$
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Invertible function: one-to-one.

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Product of elts of $T = Product of elts of S$. 
Fermat from Bijection.

**Fermat’s Little Theorem:** For prime $p$, and $a \not\equiv 0 \pmod{p}$,

\[ a^{p-1} \equiv 1 \pmod{p}. \]

**Proof:** Consider $T = \{a \cdot 1 \pmod{p}, \ldots, a \cdot (p-1) \pmod{p}\}$.

$T$ is range of function $f(x) = ax \mod (p)$ for set $S = \{1, \ldots, p-1\}$.

Invertible function: one-to-one.

$T \subseteq S$ since $0 \not\in T$.

$p$ is prime.

$\implies T = S$.

Product of elts of $T = $ Product of elts of $S$.

\[ (a \cdot 1) \cdot (a \cdot 2) \cdots (a \cdot (p-1)) \equiv 1 \cdot 2 \cdots (p-1) \pmod{p}, \]
Fermat from Bijection.

**Fermat’s Little Theorem:** For prime $p$, and $a \neq 0 \pmod{p}$,

$$a^{p-1} \equiv 1 \pmod{p}.$$  

**Proof:** Consider $T = \{a \cdot 1 \pmod{p}, \ldots, a \cdot (p-1) \pmod{p}\}$.

$T$ is range of function $f(x) = ax \pmod{p}$ for set $S = \{1, \ldots, p-1\}$.

Invertible function: one-to-one.

$T \subseteq S$ since $0 \not\in T$.

$p$ is prime.

$\implies T = S$.

Product of elts of $T = \text{Product of elts of } S$.

$$(a \cdot 1) \cdot (a \cdot 2) \cdots (a \cdot (p-1)) \equiv 1 \cdot 2 \cdots (p-1) \pmod{p},$$

Since multiplication is commutative.
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Invertible function: one-to-one.

- $T \subseteq S$ since $0 \notin T$.
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$\implies T = S$.

Product of elts of $T = \text{Product of elts of } S$.

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Each of $2, \ldots (p-1)$ has an inverse modulo $p$,
multiply by inverses to get...
\[ a^{(p-1)} \equiv 1 \pmod{p}. \]
RSA

RSA:

\[ N = p \times q \] with \( \gcd(e, (p-1)(q-1)) = 1 \).

\[ d = e - 1 \pmod{(p-1)(q-1)} \].

Theorem:

\[ x^{ed} = x \pmod{N} \]

Proof:

\[ x^{ed} - x \]

is divisible by \( p \) and \( q \) \( \Rightarrow \) theorem!

\[ x^{ed} - x = x^k(p-1)(q-1) + 1 - x = x((x^k(q-1))^{p-1}) - 1 \]

If \( x \) is divisible by \( p \), the product is.

Otherwise \( (x^k(q-1))^{p-1} = 1 \pmod{p} \) by Fermat.

\( \Rightarrow \) \((x^k(q-1))^{p-1} - 1 \) divisible by \( p \).

Similarly for \( q \).
RSA

RSA:
\[ N = p, q \]
RSA: 

\[ N = p, \ q \]

\[ e \text{ with } \gcd(e, (p - 1)(q - 1)) = 1. \]
RSA:

\[ N = p, q \]
\[ e \text{ with } \gcd(e, (p-1)(q-1)) = 1. \]
\[ d = e^{-1} \pmod{(p-1)(q-1)}. \]
RSA:

$N = p, q$

$e$ with $\gcd(e, (p - 1)(q - 1)) = 1$.

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**Theorem:** $x^{ed} = x \pmod{N}$
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\( x^{ed} - x \) is divisible by \( p \) and \( q \) \( \implies \) theorem!
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**Proof:**
\[ x^{ed} - x \text{ is divisible by } p \text{ and } q \implies \text{ theorem!} \]
\[ x^{ed} - x = x^{k(p-1)(q-1)+1} - x \]
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**Proof:**
\( x^{ed} - x \) is divisible by \( p \) and \( q \) \( \implies \) theorem!
\[ x^{ed} - x = x^{k(p-1)(q-1)+1} - x = x((x^{k(q-1)}p^{-1} - 1) \]
RSA:

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\[ e \] with \( \gcd(e, (p - 1)(q - 1)) = 1. \]
\[ d = e^{-1} \pmod{(p - 1)(q - 1)}. \]

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**Proof:**
\( x^{ed} - x \) is divisible by \( p \) and \( q \) \( \implies \) theorem!

\[ x^{ed} - x = x^{k(p-1)(q-1)+1} - x = x((x^{k(q-1)})^{p-1} - 1) \]

If \( x \) is divisible by \( p \), the product is.
RSA:

$N = p, q$

\[ e \text{ with } \gcd(e, (p - 1)(q - 1)) = 1. \]

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**Theorem:** $x^{ed} = x \pmod{N}$

**Proof:**

$x^{ed} - x$ is divisible by $p$ and $q$ $\implies$ theorem!

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Otherwise $(x^{k(q-1)})^{p-1} = 1 \pmod{p}$ by Fermat.
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If \( x \) is divisible by \( p \), the product is.
Otherwise \((x^{k(q - 1)})^{p - 1} = 1 \pmod{p}\) by Fermat.
\[ \implies (x^{k(q - 1)})^{p - 1} - 1 \text{ divisible by } p. \]

Similarly for \( q \).
RSA

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**Proof:**
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\[
  x^{ed} - x = x^{k(p-1)(q-1)+1} - x = x((x^{k(q-1)})^{p-1} - 1)
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Otherwise \( (x^{k(q-1)})^{p-1} = 1 \pmod{p} \) by Fermat.
\( \implies (x^{k(q-1)})^{p-1} - 1 \) divisible by \( p \).

Similarly for \( q \).
RSA, Public Key, and Signatures.
RSA, Public Key, and Signatures.

RSA:

N = p, q

d = e − 1 (mod (p−1)(q−1)).

Public Key Cryptography:

D(E(m, K), k) = (me)d mod N = m.

Signature scheme:

S(C) = D(C).

Announce (C, S(C))

Verify: Check C = E(C).

E(D(C, k), K) = (Cd)e = C (mod N).
RSA, Public Key, and Signatures.

RSA:
\[ N = p, q \]
RSA, Public Key, and Signatures.

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\[ N = p, q \]
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RSA, Public Key, and Signatures.

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RSA, Public Key, and Signatures.

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Public Key Cryptography:
\[ D(E(m, K), k) = (m^e)^d \pmod{N} = m. \]
RSA, Public Key, and Signatures.

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$N = p, q$

e with $\gcd(e, (p - 1)(q - 1))$.

d = $e^{-1} \pmod{(p - 1)(q - 1)}$.

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RSA, Public Key, and Signatures.

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Public Key Cryptography:

\[ D(E(m, K), k) = (m^e)^d \mod N = m. \]

Signature scheme:

\[ S(C) = D(C). \]
Announce \( (C, S(C)) \)
Verify: Check \( C = E(C). \)
RSA, Public Key, and Signatures.

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\[ N = p, q \]
\[ e \text{ with } \gcd(e, (p - 1)(q - 1)). \]
\[ d = e^{-1} \pmod{(p - 1)(q - 1)}. \]

Public Key Cryptography:
\[ D(E(m, K), k) = (m^e)^d \pmod{N} = m. \]

Signature scheme:
\[ S(C) = D(C). \]
Announce \((C, S(C))\)
Verify: Check \(C = E(C)\).
\[ E(D(C, k), K) = (C^d)^e = C \pmod{N}. \]
Midterm format

Time: 120 minutes.
Midterm format

Time: 120 minutes.
Some short answers.
Midterm format

Time: 120 minutes.

Some short answers.
  Get at ideas that you learned.
Midterm format

Time: 120 minutes.

Some short answers.
  Get at ideas that you learned.
  Know material well:
Midterm format

Time: 120 minutes.

Some short answers.
  Get at ideas that you learned.
  Know material well: fast,
Midterm format

Time: 120 minutes.

Some short answers.
  Get at ideas that you learned.
  Know material well: fast, correct.
Midterm format

Time: 120 minutes.

Some short answers.
  Get at ideas that you learned.
    Know material well: fast, correct.
    Know material medium:
Midterm format

Time: 120 minutes.

Some short answers.
  Get at ideas that you learned.
  Know material well:     fast, correct.
  Know material medium:  slower,
Midterm format

Time: 120 minutes.

Some short answers.
  Get at ideas that you learned.
  Know material well: fast, correct.
  Know material medium: slower, less correct.
Midterm format

Time: 120 minutes.

Some short answers.
Get at ideas that you learned.
  Know material well: fast, correct.
  Know material medium: slower, less correct.
  Know material not so well:
Midterm format

Time: 120 minutes.

Some short answers.
  Get at ideas that you learned.
  Know material well: fast, correct.
  Know material medium: slower, less correct.
  Know material not so well: Uh oh.
Midterm format

Time: 120 minutes.

Some short answers.
  Get at ideas that you learned.
    Know material well: fast, correct.
    Know material medium: slower, less correct.
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Some longer questions.
Midterm format

Time: 120 minutes.

Some short answers.
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Some longer questions.
  Proofs,
Midterm format

Time: 120 minutes.

Some short answers.
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    Know material medium: slower, less correct.
    Know material not so well: Uh oh.

Some longer questions.
  Proofs, algorithms,
Midterm format

Time: 120 minutes.

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  Know material well: fast, correct.
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  Know material not so well: Uh oh.

Some longer questions.
Proofs, algorithms, properties.
Midterm format

Time: 120 minutes.

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Know material well: fast, correct.
Know material medium: slower, less correct.
Know material not so well: Uh oh.

Some longer questions.
Proofs, algorithms, properties.
Not so much calculation.
Midterm format

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Will post midterm from 4 years ago to get an idea.
Midterm format

Time: 120 minutes.

Some short answers.
  Get at ideas that you learned.
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Will post midterm from 4 years ago to get an idea.
  Back when I was young
Midterm format

Time: 120 minutes.

Some short answers.

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Know material well: fast, correct.
Know material medium: slower, less correct.
Know material not so well: Uh oh.

Some longer questions.

Proofs, algorithms, properties.
Not so much calculation.

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Back when I was younger.
Midterm format

Time: 120 minutes.

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 Get at ideas that you learned.
   Know material well: fast, correct.
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 Proofs, algorithms, properties.
 Not so much calculation.

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 Back when I was younger.
Fermat/RSA

$3^6 \pmod{7}$?
$3^6 \pmod{7} = 1.$
Fermat/RSA

$3^6 \pmod{7}$? 1. Fermat: $p = 7, p - 1 = 6$
1. Fermat: $p = 7$, $p - 1 = 6$

$3^6 \pmod{7}$?  
$3^{18} \pmod{7}$?
Fermat/RSA

$3^6 \pmod{7}$? 1. Fermat: $p = 7, p - 1 = 6$

$3^{18} \pmod{7}$? 1.

$3^{60} \pmod{7}$?
Fermat/RSA

$3^6 \pmod{7}$? 1. Fermat: $p = 7$, $p - 1 = 6$

$3^{18} \pmod{7}$? 1.

$3^{60} \pmod{7}$? 1.

$3^{61} \pmod{7}$?
Fermat/RSA

$3^6 \pmod{7}$? 1. Fermat: $p = 7$, $p - 1 = 6$
$3^{18} \pmod{7}$? 1.
$3^{60} \pmod{7}$? 1.
$3^{61} \pmod{7}$? 3.
Fermat/RSA

3^6 \pmod{7}? 1. Fermat: \( p = 7, p - 1 = 6 \)
3^{18} \pmod{7}? 1.
3^{60} \pmod{7}? 1.
3^{61} \pmod{7}? 3.
2^{12} \pmod{21}?
$3^6 \pmod{7} \text{? 1. Fermat: } p = 7, \ p - 1 = 6$

$3^{18} \pmod{7} \text{? 1.}$

$3^{60} \pmod{7} \text{? 1.}$

$3^{61} \pmod{7} \text{? 3.}$

$2^{12} \pmod{21} \text{? 1.}$

$21 = (3)(7)$
$3^6 \pmod{7}$? 1. Fermat: $p = 7$, $p - 1 = 6$
$3^{18} \pmod{7}$? 1.
$3^{60} \pmod{7}$? 1.
$3^{61} \pmod{7}$? 3.
$2^{12} \pmod{21}$? 1.

$21 = (3)(7)(p - 1)(q - 1) = (2)(6) = 12$
Fermat/RSA

3^6 \pmod{7} \equiv 1. \quad \text{Fermat: } p = 7, \ p - 1 = 6

3^{18} \pmod{7} \equiv 1.

3^{60} \pmod{7} \equiv 1.

3^{61} \pmod{7} \equiv 3.

2^{12} \pmod{21} \equiv 1.

21 = (3)(7) (p - 1)(q - 1) = (2)(6) = 12

\gcd(2,12) = 1, \ x^{(p-1)(q-1)} \equiv 1 \pmod{pq}
Fermat/RSA

\[ 3^6 \pmod{7} \? 1. \quad \text{Fermat: } p = 7, \; p - 1 = 6 \]
\[ 3^{18} \pmod{7} \? 1. \]
\[ 3^{60} \pmod{7} \? 1. \]
\[ 3^{61} \pmod{7} \? 3. \]

\[ 2^{12} \pmod{21} \? 1. \]
\[ 21 = (3)(7) \; (p - 1)(q - 1) = (2)(6) = 12 \]
\[ \gcd(2, 12) = 1, \; x^{(p-1)(q-1)} = 1 \; \pmod{pq} \; 2^{12} = 1 \; \pmod{21}. \]
Fermat/RSA

$3^6 \pmod{7} \equiv 1$. Fermat: $p = 7$, $p - 1 = 6$

$3^{18} \pmod{7} \equiv 1.$

$3^{60} \pmod{7} \equiv 1.$

$3^{61} \pmod{7} \equiv 3.$

$2^{12} \pmod{21} \equiv 1.$

$21 = (3)(7) (p - 1)(q - 1) = (2)(6) = 12$

$gcd(2, 12) = 1$, $x^{(p-1)(q-1)} \equiv 1 \pmod{pq}$ $2^{12} = 1 \pmod{21}$.

$2^{14} \pmod{21}?$
3^6 \pmod{7}? 1. Fermat: \( p = 7, p - 1 = 6 \)

3^{18} \pmod{7}? 1.

3^{60} \pmod{7}? 1.

3^{61} \pmod{7}? 3.

2^{12} \pmod{21}? 1.

21 = (3)(7) (p - 1)(q - 1) = (2)(6) = 12

gcd(2, 12) = 1, x^{(p-1)(q-1)} = 1 \pmod{pq} \ 2^{12} = 1 \pmod{21}.

2^{14} \pmod{21}? 4.
3^6 \pmod{7}? 1. Fermat: \( p = 7, p - 1 = 6 \)
3^{18} \pmod{7}? 1.
3^{60} \pmod{7}? 1.
3^{61} \pmod{7}? 3.
2^{12} \pmod{21}? 1.
\[ 21 = (3)(7)(p - 1)(q - 1) = (2)(6) = 12 \]
\[ \gcd(2, 12) = 1, \quad x^{(p-1)(q-1)} = 1 \pmod{pq} \]
\[ 2^{12} = 1 \pmod{21}. \]
2^{14} \pmod{21}? 4. Technically 4 \pmod{21}. 

Wrapup.
Wrapup.

If you sent me email about Midterm conflicts
Wrapup.

If you sent me email about Midterm conflicts
Other arrangements.
Wrapup.

If you sent me email about Midterm conflicts
Other arrangements.
Should have recieved an email from me.
Wrapup.

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Other issues....
Wrapup.

If you sent me email about Midterm conflicts
Other arrangements.
Should have received an email from me.

Other issues....
satishr@cs.berkeley.edu
Wrapup.

If you sent me email about Midterm conflicts
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Other issues....
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Private message on piazza.
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