Wrapup of Polynomials.
Today.

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..and modular arithmetic.
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..and modular arithmetic.
Coutability and Uncountability.
Reed-Solomon code.

**Problem:** Communicate $n$ packets $m_1, \ldots, m_n$ on noisy channel that corrupts $\leq k$ packets.
Reed-Solomon code.

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**Reed-Solomon Code:**

1. Make a polynomial, $P(x)$ of degree $n-1$, that encodes message: coefficients, $p_0, \ldots, p_{n-1}$.
2. Send $P(1), \ldots, P(n+2k)$.

After noisy channel: Receive values $R(1), \ldots, R(n+2k)$.

**Properties:**

1. $P(i) = R(i)$ for at least $n+k$ points $i$,
2. $P(x)$ is unique degree $n-1$ polynomial that contains $\geq n+k$ received points.

Matrix view of encoding: modulo $p$.

$$
\begin{array}{c}
P(1) \\
P(2) \\
P(3) \\
\vdots \\
P(n+2k)
\end{array}
= 
\begin{array}{c}
1 \\
2 \\
3 \\
\vdots \\
(n+2k)
\end{array}
\cdot
\begin{array}{c}
p_0 \\
p_1 \\
p_2 \\
\vdots \\
p_{n-1}
\end{array}
\pmod{p}
$$
Reed-Solomon code.

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1. $P(i) = R(i)$ for at least $n+k$ points $i$. 
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Properties:

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(2) $P(x)$ is unique degree $n-1$ polynomial
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**After noisy channel:** Recieve values $R(1), \ldots, R(n+2k)$.

**Properties:**

1. $P(i) = R(i)$ for at least $n + k$ points $i$,
2. $P(x)$ is unique degree $n - 1$ polynomial that contains $\geq n + k$ received points.
Reed-Solomon code.

**Problem:** Communicate \( n \) packets \( m_1, \ldots, m_n \) on noisy channel that corrupts \( \leq k \) packets.

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Matrix view of encoding: modulo \( p \).
Reed-Solomon code.

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**Matrix view of encoding:** modulo \( p \).

\[
\begin{bmatrix}
P(1) \\
P(2) \\
P(3) \\
\vdots \\
P(n+2k)
\end{bmatrix}
= \begin{bmatrix}
1 & 1 & 1^2 & \cdots & 1^{2^{n-1}} \\
1 & 2 & 2^2 & \cdots & 2^{n-1} \\
1 & 3 & 3^2 & \cdots & 3^{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & (n+2k) & (n+2k)^2 & \cdots & (n+2k)^{n-1}
\end{bmatrix}
\begin{bmatrix}
p_0 \\
p_1 \\
p_2 \\
\vdots \\
p_{n-1}
\end{bmatrix}
\pmod{p}
\]
Berlekamp-Welsh Algorithm

\[ P(x): \text{degree } n - 1 \text{ polynomial.} \]
Berlekamp-Welsh Algorithm

$P(x)$: degree $n - 1$ polynomial.
Send $P(1), \ldots, P(n + 2k)$
Berlekamp-Welsh Algorithm

\( P(x) \): degree \( n - 1 \) polynomial.
Send \( P(1), \ldots, P(n + 2k) \)
Receive \( R(1), \ldots, R(n + 2k) \)
Berlekamp-Welsh Algorithm

$P(x)$: degree $n - 1$ polynomial.
Send $P(1), \ldots, P(n + 2k)$
Receive $R(1), \ldots, R(n + 2k)$
At most $k$ i’s where $P(i) \neq R(i)$. 

Idea:
$E(x)$ is error locator polynomial.
Root at each error point.
Degree $k$.
$Q(x) = P(x)E(x)$ or degree $n + k - 1$ polynomial.
Set up system corresponding to $Q(i) = R(i)E(i)$ where $Q(x)$ is degree $n + k - 1$ polynomial.
Coefficients: $a_0, \ldots, a_{n + k - 1}$,
$E(x)$ degree $k$ polynomial.
Coefficients: $b_0, \ldots, b_{k - 1}, 1$.
Matrix equations: modulo $p$

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & 2 & \cdots & 2 \\
\vdots & \vdots & \ddots & \vdots \\
1 & n + 2k & \cdots & n + 2k \\
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
\vdots \\
a_{n + k - 1} \\
\end{bmatrix}
=
\begin{bmatrix}
R(1) \\
R(2) \\
\vdots \\
R(n + 2k) \\
\end{bmatrix}
\begin{bmatrix}
b_0 \\
b_1 \\
\vdots \\
b_{k - 1} \\
1 \\
\end{bmatrix}
\]

Solve.

Then output $P(x) = Q(x)/E(x)$. 

Berlekamp-Welsh Algorithm

$P(x)$: degree $n - 1$ polynomial.
Send $P(1), \ldots, P(n + 2k)$
Received $R(1), \ldots, R(n + 2k)$
At most $k$ $i$’s where $P(i) \neq R(i)$.
Berlekamp-Welsh Algorithm

\[ P(x) : \text{degree } n - 1 \text{ polynomial.} \]
Send \( P(1), \ldots, P(n + 2k) \)
Receive \( R(1), \ldots, R(n + 2k) \)
At most \( k \) \( i \)'s where \( P(i) \neq R(i) \).

Idea:

\[ E(x) \text{ is error locator polynomial.} \]
\[ \text{Root at each error point.} \]
\[ Q(x) = P(x)E(x) \text{ or degree } n + k - 1 \text{ polynomial.} \]
Set up system corresponding to \( Q(x)_i = R(x)_i E(x)_i \) where \( Q(x) \) is degree \( n + k - 1 \) polynomial.
Coefficients: \( a_0, \ldots, a_{n + k - 1} \)
Coefficients: \( b_0, \ldots, b_{k - 1} \), 1
Matrix equations: modulo \( p \)
\[
\begin{bmatrix}
1 & 1 & & & & & \cdot & \cdot & \cdot \\
1 & 2 & & & & & \cdot & \cdot & \\
1 & 3 & & & & & \cdot & \cdot & \\
\vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & (n + 2k) & & & & & \cdot & \cdot & \\
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
\vdots \\
a_{n + k - 1} \\
\end{bmatrix}
= \begin{bmatrix}
R(1)_0 & 0 & & & & & \cdot & \cdot & 0 \\
0 & 0 & & & & & \cdot & \cdot & \cdot \\
0 & 0 & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{bmatrix}
\begin{bmatrix}
b_0 \\
b_1 \\
b_2 \\
\vdots \\
b_{k - 1} \\
1 \\
\end{bmatrix}
\]
Solve.
Then output \( P(x) = Q(x) / E(x) \).
Berlekamp-Welsh Algorithm

\( P(x) \): degree \( n - 1 \) polynomial.
Send \( P(1), \ldots, P(n + 2k) \)
Receive \( R(1), \ldots, R(n + 2k) \)
At most \( k \) i’s where \( P(i) \neq R(i) \).

Idea:
\( E(x) \) is error locator polynomial.
Berlekamp-Welsh Algorithm

\( P(x) \): degree \( n - 1 \) polynomial.
Send \( P(1), \ldots, P(n + 2k) \)
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At most \( k \) i’s where \( P(i) \neq R(i) \).

Idea:
\( E(x) \) is error locator polynomial.
   Root at each error point.
Berlekamp-Welsh Algorithm

$P(x)$: degree $n - 1$ polynomial.
Send $P(1), \ldots, P(n + 2k)$
Receive $R(1), \ldots, R(n + 2k)$
At most $k$ $i$'s where $P(i) \neq R(i)$.

Idea:
$E(x)$ is error locator polynomial.
  Root at each error point. Degree $k$.
$Q(x) = P(x)E(x)$ or degree $n + k - 1$ polynomial.
Berlekamp-Welsh Algorithm

$P(x)$: degree $n - 1$ polynomial.
Send $P(1), \ldots, P(n + 2k)$
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$E(x)$ is error locator polynomial.
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$Q(x) = P(x)E(x)$ or degree $n + k - 1$ polynomial.

Set up system corresponding to $Q(i) = R(i)E(i)$ where
Berlekamp-Welsh Algorithm

$P(x)$: degree $n - 1$ polynomial.
Send $P(1), \ldots, P(n + 2k)$
Receive $R(1), \ldots, R(n + 2k)$
At most $k$ i’s where $P(i) \neq R(i)$.

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$E(x)$ is error locator polynomial.
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Berlekamp-Welsh Algorithm

\[ P(x) : \text{degree } n - 1 \text{ polynomial.} \]
Send \( P(1), \ldots, P(n+2k) \)
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At most \( k \) i’s where \( P(i) \neq R(i) \).

Idea:
\( E(x) \) is error locator polynomial.
  Root at each error point. Degree \( k \).
\[ Q(x) = P(x)E(x) \text{ or degree } n + k - 1 \text{ polynomial.} \]
Set up system corresponding to \( Q(i) = R(i)E(i) \) where
  \( Q(x) \) is degree \( n + k - 1 \) polynomial. Coefficients: \( a_0, \ldots, a_{n+k-1} \)
Berlekamp-Welsh Algorithm

\( P(x) \): degree \( n - 1 \) polynomial.
Send \( P(1), \ldots, P(n + 2k) \)
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At most \( k \) \( i \)'s where \( P(i) \neq R(i) \).

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\( E(x) \) is error locator polynomial.
Root at each error point. Degree \( k \).
\( Q(x) = P(x)E(x) \) or degree \( n + k - 1 \) polynomial.

Set up system corresponding to \( Q(i) = R(i)E(i) \) where
\( Q(x) \) is degree \( n + k - 1 \) polynomial. Coefficients: \( a_0, \ldots, a_{n+k-1} \)
\( E(x) \) is degree \( k \) polynomial.
Berlekamp-Welsh Algorithm

\[ P(x): \text{degree } n - 1 \text{ polynomial.} \]
Send \( P(1), \ldots, P(n + 2k) \)
Receive \( R(1), \ldots, R(n + 2k) \)
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\( E(x) \) is error locator polynomial.
Root at each error point. Degree \( k \).
\( Q(x) = P(x)E(x) \) or degree \( n + k - 1 \) polynomial.

Set up system corresponding to \( Q(i) = R(i)E(i) \) where
\( Q(x) \) is degree \( n + k - 1 \) polynomial. Coefficients: \( a_0, \ldots, a_{n+k-1} \)
\( E(x) \) is degree \( k \) polynomial. Coefficients: \( b_0, \ldots, b_{k-1}, 1 \)
Berlekamp-Welsh Algorithm

\( P(x) \): degree \( n - 1 \) polynomial.
Send \( P(1), \ldots, P(n+2k) \)
Receive \( R(1), \ldots, R(n+2k) \)
At most \( k \) i's where \( P(i) \neq R(i) \).

Idea:
\( E(x) \) is error locator polynomial.
  Root at each error point. Degree \( k \).
\( Q(x) = P(x)E(x) \) or degree \( n + k - 1 \) polynomial.

Set up system corresponding to \( Q(i) = R(i)E(i) \) where
  \( Q(x) \) is degree \( n + k - 1 \) polynomial. Coefficients: \( a_0, \ldots, a_{n+k-1} \)
  \( E(x) \) is degree \( k \) polynomial. Coefficients: \( b_0, \ldots, b_{k-1}, 1 \)

Matrix equations: modulo \( p \)!
Berlekamp-Welsh Algorithm

\[ P(x) : \text{degree } n - 1 \text{ polynomial.} \]
Send \( P(1), \ldots, P(n+2k) \)
Receive \( R(1), \ldots, R(n+2k) \)
At most \( k \) i’s where \( P(i) \neq R(i) \).

Idea:
\( E(x) \) is error locator polynomial.
Root at each error point. Degree \( k \).
\( Q(x) = P(x)E(x) \) or degree \( n + k - 1 \) polynomial.

Set up system corresponding to \( Q(i) = R(i)E(i) \) where
\( Q(x) \) is degree \( n + k - 1 \) polynomial. Coefficients: \( a_0, \ldots, a_{n+k-1} \)
\( E(x) \) is degree \( k \) polyonimial. Coefficients: \( b_0, \ldots, b_{k-1}, 1 \)

Matrix equations: modulo \( p \!

\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & 2 & \cdots & 2^{n+k-1} \\
1 & 3 & \cdots & 3^{n+k-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & (n+2k) & \cdots & (n+2k)^{n+k-1}
\end{pmatrix}
\begin{pmatrix}
a_0 \\
a_1 \\
a_2 \\
\vdots \\
a_{n+k-1}
\end{pmatrix}
= 
\begin{pmatrix}
R(1) & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & R(n+2k)
\end{pmatrix}
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & 2 & \cdots & 2^k \\
1 & 3 & \cdots & 3^k \\
\vdots & \vdots & \ddots & \vdots \\
1 & (n+2k) & \cdots & (n+2k)^k
\end{pmatrix}
\begin{pmatrix}
b_0 \\
b_1 \\
b_2 \\
\vdots \\
b_{k-1}
\end{pmatrix}
\]
Berlekamp-Welsh Algorithm

\( P(x) \): degree \( n-1 \) polynomial.

Send \( P(1), \ldots, P(n+2k) \)

Receive \( R(1), \ldots, R(n+2k) \)

At most \( k \) i's where \( P(i) \neq R(i) \).

Idea:
\( E(x) \) is error locator polynomial.

Root at each error point. Degree \( k \).

\( Q(x) = P(x)E(x) \) or degree \( n+k-1 \) polynomial.

Set up system corresponding to \( Q(i) = R(i)E(i) \) where
\( Q(x) \) is degree \( n+k-1 \) polynomial. Coefficients: \( a_0, \ldots, a_{n+k-1} \)
\( E(x) \) is degree \( k \) polynomial. Coefficients: \( b_0, \ldots, b_{k-1}, 1 \)

Matrix equations: modulo \( p! \)

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & 2 & \cdots & 2^{n+k-1} \\
1 & 3 & \cdots & 3^{n+k-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & (n+2k) & \cdots & (n+2k)^{n+k-1}
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
\vdots \\
a_{n+k-1}
\end{bmatrix}
= 
\begin{bmatrix}
R(1) & \cdots & 0 \\
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & R(n+2k)
\end{bmatrix}
\begin{bmatrix}
1 & 1^k \\
1 & 2^k \\
\vdots \\
1 & (n+2k)^{n+k-1}
\end{bmatrix}
\begin{bmatrix}
b_0 \\
b_1 \\
\vdots \\
b_{k-1}
\end{bmatrix}
\]

Solve.
Berlekamp-Welsh Algorithm

\( P(x) \): degree \( n - 1 \) polynomial.
Send \( P(1), \ldots, P(n+2k) \)
Receive \( R(1), \ldots, R(n+2k) \)
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Matrix equations: modulo \( p! \)

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & 2 & \cdots & 2^{n+k-1} \\
1 & 3 & \cdots & 3^{n+k-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & (n+2k) & \cdots & (n+2k)^{n+k-1}
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
\vdots \\
a_{n+k-1}
\end{bmatrix}
= 
\begin{bmatrix}
R(1) & \cdots & 0 \\
0 & \cdots & 0 \\
0 & \cdots & R(n+2k) \\
\vdots & \ddots & \vdots \\
1 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
1 \\
\vdots \\
1
\end{bmatrix}
\begin{bmatrix}
1 & 2^k \\
1 & 3^k \\
\vdots \\
1 & (n+2k)^k \\
1
\end{bmatrix}
\begin{bmatrix}
b_0 \\
b_1 \\
\vdots \\
b_{k-1} \\
1
\end{bmatrix}
\]

Solve. Then output \( P(x) = Q(x)/E(x) \).
Berlekamp-Welsh algorithm decodes correctly when $k$ errors!
Summary: polynomials.
Set of $d + 1$ points determines degree $d$ polynomial.
Summary: polynomials.

Set of $d + 1$ points determines degree $d$ polynomial.

Encode secret using degree $k - 1$ polynomial:
Summary: polynomials.

Set of \( d + 1 \) points determines degree \( d \) polynomial.

Encode secret using degree \( k - 1 \) polynomial:
- Can share with \( n \) people.
Summary: polynomials.

Set of $d + 1$ points determines degree $d$ polynomial.

Encode secret using degree $k - 1$ polynomial:
    Can share with $n$ people. Any $k$ can recover!
Summary: polynomials.

Set of $d + 1$ points determines degree $d$ polynomial.

Encode secret using degree $k - 1$ polynomial:
   Can share with $n$ people. Any $k$ can recover!

Encode message using degree $n - 1$ polynomial:
Summary: polynomials.

Set of $d + 1$ points determines degree $d$ polynomial.

Encode secret using degree $k - 1$ polynomial:
  Can share with $n$ people. Any $k$ can recover!

Encode message using degree $n - 1$ polynomial:
  $n$ packets of information.
Summary: polynomials.

Set of $d+1$ points determines degree $d$ polynomial.

Encode secret using degree $k-1$ polynomial:
   Can share with $n$ people. Any $k$ can recover!

Encode message using degree $n-1$ polynomial:
   $n$ packets of information.

Send $n+k$ packets (point values).
Summary: polynomials.
Set of $d + 1$ points determines degree $d$ polynomial.

Encode secret using degree $k - 1$ polynomial:
   Can share with $n$ people. Any $k$ can recover!

Encode message using degree $n - 1$ polynomial:
   $n$ packets of information.

Send $n + k$ packets (point values).
   Can recover from $k$ losses:
Summary: polynomials.

Set of $d + 1$ points determines degree $d$ polynomial.

Encode secret using degree $k – 1$ polynomial:
  Can share with $n$ people. Any $k$ can recover!

Encode message using degree $n – 1$ polynomial:
  $n$ packets of information.

Send $n + k$ packets (point values).
  Can recover from $k$ losses: Still have $n$ points!
Summary: polynomials.

Set of $d + 1$ points determines degree $d$ polynomial.

Encode secret using degree $k - 1$ polynomial:
   Can share with $n$ people. Any $k$ can recover!

Encode message using degree $n - 1$ polynomial:
   $n$ packets of information.

Send $n + k$ packets (point values).
   Can recover from $k$ losses: Still have $n$ points!

Send $n + 2k$ packets (point values).
Summary: polynomials.

Set of $d + 1$ points determines degree $d$ polynomial.

Encode secret using degree $k - 1$ polynomial:
   Can share with $n$ people. Any $k$ can recover!

Encode message using degree $n - 1$ polynomial:
   $n$ packets of information.

Send $n + k$ packets (point values).
   Can recover from $k$ losses: Still have $n$ points!

Send $n + 2k$ packets (point values).
   Can recover from $k$ corruptionss.
Summary: polynomials.

Set of \( d + 1 \) points determines degree \( d \) polynomial.

Encode secret using degree \( k - 1 \) polynomial:
  Can share with \( n \) people. Any \( k \) can recover!

Encode message using degree \( n - 1 \) polynomial:
  \( n \) packets of information.

Send \( n + k \) packets (point values).
  Can recover from \( k \) losses: Still have \( n \) points!

Send \( n + 2k \) packets (point values).
  Can recover from \( k \) corruptionss.
  Only one polynomial contains \( n + k \)
Summary: polynomials.

Set of \(d + 1\) points determines degree \(d\) polynomial.

Encode secret using degree \(k - 1\) polynomial:
  Can share with \(n\) people. Any \(k\) can recover!

Encode message using degree \(n - 1\) polynomial:
  \(n\) packets of information.

Send \(n + k\) packets (point values).
  Can recover from \(k\) losses: Still have \(n\) points!

Send \(n + 2k\) packets (point values).
  Can recover from \(k\) corruptionss.
    Only one polynomial contains \(n + k\)
  Efficiency.
Summary: polynomials.

Set of $d + 1$ points determines degree $d$ polynomial.

Encode secret using degree $k - 1$ polynomial:
  Can share with $n$ people. Any $k$ can recover!

Encode message using degree $n - 1$ polynomial:
  $n$ packets of information.

Send $n + k$ packets (point values).
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Error Locator Polynomial.
Summary: polynomials.
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Efficiency.
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Relations:
Summary: polynomials.

Set of $d+1$ points determines degree $d$ polynomial.

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Encode message using degree $n-1$ polynomial:
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Send $n+k$ packets (point values).
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Send $n+2k$ packets (point values).
  Can recover from $k$ corruptionss.
    Only one polynomial contains $n+k$

Efficiency.
  Magic!!!!
  Error Locator Polynomial.

Relations:
  Linear code.
Summary: polynomials.

Set of \( d + 1 \) points determines degree \( d \) polynomial.

Encode secret using degree \( k - 1 \) polynomial:
- Can share with \( n \) people. Any \( k \) can recover!

Encode message using degree \( n - 1 \) polynomial:
- \( n \) packets of information.

Send \( n + k \) packets (point values).
- Can recover from \( k \) losses: Still have \( n \) points!

Send \( n + 2k \) packets (point values).
- Can recover from \( k \) corruptions.
  - Only one polynomial contains \( n + k \)

Efficiency.
- Magic!!!!
- Error Locator Polynomial.

Relations:
- Linear code.
- Almost any coding matrix works.
Summary: polynomials.

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Efficiency.
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Relations:
  Linear code.
    Almost any coding matrix works.
  Vandermonde matrix (the one for Reed-Solomon)
Summary: polynomials.

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Encode secret using degree $k - 1$ polynomial:
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Efficiency.
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  Error Locator Polynomial.

Relations:
  Linear code.
  Almost any coding matrix works.
  Vandermonde matrix (the one for Reed-Solomon) allows for efficiency.
Summary: polynomials.
Set of $d + 1$ points determines degree $d$ polynomial.

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Efficiency.
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Relations:
   Linear code.
   Almost any coding matrix works.
   Vandermonde matrix (the one for Reed-Solomon)
   allows for efficiency. Magic of polynomials.
Summary: polynomials.

Set of $d + 1$ points determines degree $d$ polynomial.

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Efficiency.
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  Error Locator Polynomial.

Relations:
  Linear code.
    Almost any coding matrix works.
    Vandermonde matrix (the one for Reed-Solomon).
      allows for efficiency. Magic of polynomials.
  Other Algebraic-Geometric codes.
Wrapping up: RSA example with “easy” extended gcd.

Example: $p = 7$, $q = 11$. 
Wrapping up: RSA example with “easy” extended gcd.

Example: \( p = 7, \ q = 11. \)

\( N = 77. \)
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$(p - 1)(q - 1) = 60$
Wrapping up: RSA example with “easy” extended gcd.

Example: $p = 7$, $q = 11$.

$N = 77$.

$(p - 1)(q - 1) = 60$

Choose $e = 7$, since $\gcd(7, 60) = 1$. 

\[
\begin{align*}
7 & (0) + 60 (1) = 60 \\
7 & (1) + 60 (0) = 7 \\
7 & (−8) + 60 (1) = 4 \\
7 & (9) + 60 (−1) = 3 \\
7 & (−17) + 60 (2) = 1
\end{align*}
\]

Confirm: $−119 + 120 = 1$

$d = e - 1 = −17 = 43 \pmod{60}$
Example: $p = 7$, $q = 11$.

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$\operatorname{egcd}(7, 60)$. 
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egcd(7,60).

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\begin{align*}
7(0) + 60(1) & = 60 \\
7(1) + 60(0) & = 7
\end{align*}
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Wrapping up: RSA example with “easy” extended gcd.

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\( (p - 1)(q - 1) = 60 \)
Choose \( e = 7, \) since \( \gcd(7, 60) = 1. \)
\[ \text{egcd}(7,60). \]

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\end{align*}

Confirm: \(-119 + 120 = 1\)
\(d = e^{-1} = -17 = 43 \equiv (\mod 60)\)
Farewell (for now) to modular arithmetic...

Modular arithmetic modulo a prime.

4 > 3 ?

Yes!

4 > 3 (mod 7) ?

Yes...

maybe?

−3 > 3 (mod 7) ?

Uh oh..

−3 = 4 (mod 7).

Another problem.

4 is close to 3.

But can you get closer?

Sure.

3.5.

Closer.

Sure?

3.25, 3.1, 3.000001. ...

For real numbers we have the notion of limit, continuity, and derivative. ....and Calculus.

For modular arithmetic...

no Calculus.

Sad face!
Farewell (for now) to modular arithmetic...

Modular arithmetic modulo a prime.

Add, subtract, commutative, associative,
Farewell (for now) to modular arithmetic...

Modular arithmetic modulo a prime.

Add, subtract, commutative, associative, inverses!
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Modular arithmetic modulo a prime.

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Allow for solving linear systems, discussing polynomials...
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Why not modular arithmetic all the time?
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Modular arithmetic modulo a prime.

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Why not modular arithmetic all the time?

$4 \not> 3$ ?
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Why not modular arithmetic all the time?

$4 \n 3 \text{ ? Yes!}$
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Add, subtract, commutative, associative, inverses!
Allow for solving linear systems, discussing polynomials...

Why not modular arithmetic all the time?

\[ 4 > 3 \, ? \, Yes! \]

\[ 4 > 3 \, (\text{mod } 7)\, ? \, Yes...maybe? \]

\[ -3 > 3 \, (\text{mod } 7)\, ? \, Uh \, oh.. \, -3 = 4 \, (\text{mod } 7). \]

Another problem.

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But can you get closer?
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But can you get closer? Sure. 3.5. Closer. Sure?
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Modular arithmetic modulo a prime.
   Add, subtract, commutative, associative, inverses!
   Allow for solving linear systems, discussing polynomials...

Why not modular arithmetic all the time?
   \( 4 \equiv 3 \pmod{7} \) ? Yes!
   \( 4 \equiv 3 \pmod{7} ? \) Yes...maybe?
   \(-3 \equiv 3 \pmod{7} \) ? Uh oh.. \(-3 = 4 \pmod{7}\).

Another problem.

4 is close to 3.
But can you get closer? Sure. 3.5. Closer. Sure? 3.25,
Farewell (for now) to modular arithmetic...

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   Add, subtract, commutative, associative, inverses!
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Farewell (for now) to modular arithmetic...

Modular arithmetic modulo a prime.

Add, subtract, commutative, associative, inverses!
Allow for solving linear systems, discussing polynomials...

Why not modular arithmetic all the time?

4 \geq 3 \ ? \ Yes!

4 \geq 3 \ (\text{mod} \ 7) \ ? \ Yes...maybe?

-3 > 3 \ (\text{mod} \ 7) \ ? \ Uh \ oh.. \ -3 = 4 \ (\text{mod} \ 7).

Another problem.

4 is close to 3.
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For reals numbers we have the notion of limit, continuity, and \textit{derivative}......
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For reals numbers we have the notion of limit, continuity, and derivative.......

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Add, subtract, commutative, associative, inverses!
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For reals numbers we have the notion of limit, continuity, and derivative.......

....and Calculus.

For modular arithmetic...no Calculus. Sad face!
Next up: how big is infinity.
Next up: how big is infinity.

- Countable
- Countably infinite.
- Enumeration
How big are the reals or the integers?

Infinite!
How big are the reals or the integers?

Infinite!

Is one bigger or smaller?
Same size?

Make a function $f$: Circles $\rightarrow$ Squares.

- $f(\text{red circle}) = \text{red square}$
- $f(\text{blue circle}) = \text{blue square}$
- $f(\text{circle with black border}) = \text{square with black border}$

One to one: Each circle mapped to different square.

Onto: Each square mapped from some circle.

Isomorphism principle: If there is $f: D \rightarrow R$ that is one to one and onto, then, $|D| = |R|$. 
Same size?

- Red circle
- Blue circle
- Green circle
- Red square
- Blue square
- Green square

Same number?

Make a function \( f \):

- Circles \( \rightarrow \) Squares.

\( f(\) red circle\) = red square

\( f(\) blue circle\) = blue square

\( f(\) circle with black border\) = square with black border

One to one.

Each circle mapped to different square.

One to One: For all \( x, y \in D \), \( x \neq y \implies f(x) \neq f(y) \).

Onto.

Each square mapped to from some circle.

Onto: For all \( s \in R \), \( \exists c \in D, s = f(c) \).

Isomorphism principle:

If there is \( f : D \rightarrow R \) that is one to one and onto, then, \(|D| = |R|\).
Same size?

Same number?
Make a function $f : \text{Circles} \rightarrow \text{Squares}$. 

Isomorphism principle: If there is $f : D \rightarrow R$ that is one to one and onto, then, $|D| = |R|$. 
Same size?

Make a function $f : \text{Circles} \rightarrow \text{Squares}$. 

$f(\text{red circle}) = \text{red square}$
Same size?

Same number?
Make a function $f : \text{Circles} \rightarrow \text{Squares}$.
$f(\text{red circle}) = \text{red square}$
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Same number?

One to one: For all $x, y \in D$, $x \neq y \Rightarrow f(x) \neq f(y)$.

Onto: For all $s \in R$, $\exists c \in D$, $s = f(c)$.

Isomorphism principle: If there is $f : D \rightarrow R$ that is one to one and onto, then, $|D| = |R|$. 
Same size?

Same number?
Make a function $f : \text{Circles} \rightarrow \text{Squares}$.

$f(\text{red circle}) = \text{red square}$

$f(\text{blue circle}) = \text{blue square}$

$f(\text{circle with black border}) = \text{square with black border}$

One to one.
Same size?

Make a function \( f : \text{Circles} \rightarrow \text{Squares} \).
\[
f(\text{red circle}) = \text{red square} \\
f(\text{blue circle}) = \text{blue square} \\
f(\text{circle with black border}) = \text{square with black border}
\]
One to one. Each circle mapped to different square.

Same number?

One to One: For all \( x, y \in D \), \( x \neq y \) \( \Rightarrow \) \( f(x) \neq f(y) \).

Onto.

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One to One: For all \( x, y \in D \), \( x \neq y \implies f(x) \neq f(y) \).

Onto. Each square mapped to from some circle.
Same size?

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Isomorphism principle.

Given a function, \( f : D \rightarrow R \).
Isomorphism principle.

Given a function, $f : D \rightarrow R$.

**One to One:**
Isomorphism principle.

Given a function, \( f : D \rightarrow R \).

**One to One:**
For all \( \forall x, y \in D, x \neq y \implies f(x) \neq f(y) \).

**Onto:**
For all \( \forall y \in R, \exists x \in D, y = f(x) \).

\( f(\cdot) \) is a bijection if it is one to one and onto.

Isomorphism principle:
If there is a bijection \( f : D \rightarrow R \) then \( |D| = |R| \).
Given a function, \( f : D \rightarrow R \).

**One to One:**
For all \( \forall x, y \in D, x \neq y \implies f(x) \neq f(y) \).

or

**Onto:**
For all \( \forall y \in R, \exists x \in D, y = f(x) \).

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Isomorphism principle.

Given a function, $f : D \to R$.

**One to One:**
For all $\forall x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.

or

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Given a function, \( f : D \rightarrow R \).

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or
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**Onto:** For all \( y \in R, \exists x \in D, y = f(x) \).
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**Onto:** For all \( y \in R, \exists x \in D, y = f(x) \).

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Isomorphism principle.

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**Isomorphism principle:**
If there is a bijection \( f : D \rightarrow R \) then \( |D| = |R| \).
Countable.

How to count?
How to count?
0,
Countable.

How to count?
0, 1,
Countable.

How to count?
0, 1, 2,
Countable.

How to count?
0, 1, 2, 3,
Countable.

How to count?

0, 1, 2, 3, …
How to count?
0, 1, 2, 3, …
The Counting numbers.
Countable.

How to count?
0, 1, 2, 3, …
The Counting numbers.
The natural numbers! \( N \)
How to count?
0, 1, 2, 3, …
The Counting numbers.
The natural numbers! $N$

Definition: $S$ is **countable** if there is a bijection between $S$ and some subset of $N$. 
How to count?
0, 1, 2, 3, ...

The Counting numbers.
The natural numbers! \(N\)

Definition: \(S\) is **countable** if there is a bijection between \(S\) and some subset of \(N\).

If the subset of \(N\) is finite, \(S\) has finite **cardinality**.
Countable.

How to count?
0, 1, 2, 3, ...

The Counting numbers.
The natural numbers! \( N \)

Definition: \( S \) is **countable** if there is a bijection between \( S \) and some subset of \( N \).

If the subset of \( N \) is finite, \( S \) has finite **cardinality**.

If the subset of \( N \) is infinite, \( S \) is **countably infinite**.
Where’s 0?

Which is bigger?

Natural numbers. 0, 1, 2, 3, ...

Positive integers. 1, 2, 3, ...

Where’s 0?

More natural numbers!

Consider $f(z) = z - 1$.

For any two $z_1 \neq z_2 \Rightarrow z_1 - 1 \neq z_2 - 1 \Rightarrow f(z_1) \neq f(z_2)$.

One to one!

For any natural number $n$, for $z = n + 1$, $f(z) = (n + 1) - 1 = n$.

Onto for $\mathbb{N}$ Bijection! $\Rightarrow |\mathbb{Z}^+| = |\mathbb{N}|$.

But.. but Where’s zero?

"Comes from 1."
Where’s 0?

Which is bigger?
The positive integers, \( \mathbb{Z}^+ \), or the natural numbers, \( \mathbb{N} \).
Where’s 0?

Which is bigger?
The positive integers, \( \mathbb{Z}^+ \), or the natural numbers, \( \mathbb{N} \).

Natural numbers. 0,
Where’s 0?

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Natural numbers. 0, 1,
Where’s 0?

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Natural numbers. 0, 1, 2,
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Positive integers. 1,
Where’s 0?

Which is bigger?
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Natural numbers. 0, 1, 2, 3, ....
Positive integers. 1, 2,
Where’s 0?

Which is bigger?
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Natural numbers. 0, 1, 2, 3, ….

Positive integers. 1, 2, 3,
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One to one!

For any natural number $n$, for $z = n + 1$, $f(z) = (n + 1) - 1$
Where’s 0?

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Bijection! \( \implies |\mathbb{Z}^+| = |\mathbb{N}| \).
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But.. but
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Onto for \( \mathbb{N} \)

Bijection! \( \implies |\mathbb{Z}^+| = |\mathbb{N}| \).

But.. but Where’s zero? “Comes from 1.”
A bijection is a bijection.
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Notice that there is a bijection between $N$ and $Z^+$ as well.
A bijection is a bijection.

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A bijection is a bijection.

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A bijection is a bijection.

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$f(n) = n + 1$. 0 → 1, 1 → 2, …
A bijection is a bijection.

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$$f(n) = n + 1. \ 0 \rightarrow 1, \ 1 \rightarrow 2, \ldots$$

Bijection from $A$ to $B \iff$ a bijection from $B$ to $A$. 

Inverse function! Can prove equivalence either way. 

Bijection to or from natural numbers implies countably infinite.
A bijection is a bijection.

Notice that there is a bijection between $N$ and $Z^+$ as well.
$f(n) = n + 1$. $0 \rightarrow 1, 1 \rightarrow 2, \ldots$

Bijection from $A$ to $B \implies$ a bijection from $B$ to $A$. 
A bijection is a bijection.

Notice that there is a bijection between $N$ and $Z^+$ as well. $f(n) = n + 1$. $0 \rightarrow 1, 1 \rightarrow 2, \ldots$

Bijection from $A$ to $B \implies$ a bijection from $B$ to $A$.

Inverse function!
A bijection is a bijection.

Notice that there is a bijection between $N$ and $\mathbb{Z}^+$ as well.

$$f(n) = n + 1. \ 0 \rightarrow 1, \ 1 \rightarrow 2, \ldots$$

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Can prove equivalence either way.
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Notice that there is a bijection between $\mathbb{N}$ and $\mathbb{Z}^+$ as well.

$f(n) = n + 1$. $0 \rightarrow 1, 1 \rightarrow 2, \ldots$

Bijection from $A$ to $B \implies$ a bijection from $B$ to $A$.

Inverse function!

Can prove equivalence either way.
Bijection to or from natural numbers implies countably infinite.
More large sets.

**E** - Even natural numbers?
More large sets.

$E$ - Even natural numbers?

$f : N \rightarrow E$. 
More large sets.

$E$ - Even natural numbers?

$f : N \rightarrow E$.

$f(n) \rightarrow 2n$. 
More large sets.

$E$ - Even natural numbers?

$f : N \rightarrow E$.

$f(n) \rightarrow 2n$.

Onto:
More large sets.

$E$ - Even natural numbers?

$f : N \rightarrow E$.

$f(n) \rightarrow 2n$.

Onto: $\forall e \in E$, $f(e/2) = e$. 

Evens are countably infinite.

Evens are same size as all natural numbers.
More large sets.

$E$ - Even natural numbers?

$f : N \rightarrow E$.

$f(n) \rightarrow 2n$.

Onto: $\forall e \in E, f(e/2) = e$. $e/2$ is natural since $e$ is even
More large sets.

\( E \) - Even natural numbers?

\( f : N \to E. \)

\( f(n) \to 2n. \)

Onto: \( \forall e \in E, f(e/2) = e. \) \( e/2 \) is natural since \( e \) is even

One-to-one:
More large sets.

$E$ - Even natural numbers?

$f : N \rightarrow E$.

$f(n) \rightarrow 2n$.

Onto: $\forall e \in E, f(e/2) = e$. $e/2$ is natural since $e$ is even

One-to-one: $\forall x, y \in N, x \neq y \implies 2x \neq 2y$. 

Evens are countably infinite.

Evens are same size as all natural numbers.
$E$ - Even natural numbers?

$f : N \rightarrow E$.

$f(n) \rightarrow 2n$.

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One-to-one: $\forall x, y \in N, x \neq y \implies 2x \neq 2y$. $\equiv f(x) \neq f(y)$
More large sets.

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Evens are countably infinite.
E - Even natural numbers?

$$f : N \rightarrow E.$$  

$$f(n) \rightarrow 2n.$$  

Onto: $$\forall e \in E, f(e/2) = e.$$  
e/2 is natural since e is even

One-to-one: $$\forall x, y \in N, x \neq y \implies 2x \neq 2y. \equiv f(x) \neq f(y)$$

Evens are countably infinite.
Evens are same size as all natural numbers.
All integers?

What about Integers, \( \mathbb{Z} \)?
All integers?

What about Integers, $\mathbb{Z}$?
Define $f : \mathbb{N} \rightarrow \mathbb{Z}$.

$$f(n) = \begin{cases} 
\frac{n}{2} & \text{if } n \text{ even} \\
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All integers?

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Define $f : \mathbb{N} \rightarrow \mathbb{Z}$.

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Integers and naturals have same size!
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Notice that: A listing “is” a bijection with a subset of natural numbers.
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Function ≡ “Position in list.”
If finite: bijection with \( \{0, \ldots, |S| - 1\} \)
If infinite: bijection with \( \mathbb{N} \).
Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.
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“Output element of $S$”,

$Z = \{0, 1, -1, 2, -2, ....\}$

When do you get to $-1$ at infinity?

Need to be careful.

$-$ streams!
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61A
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61A --- streams!
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$\mathbb{Z}^+$ is countable.
It is infinite since the list goes on.
There is a bijection with the natural numbers.
So it is countably infinite.
All countably infinite sets have the same cardinality.
Countably infinite subsets.

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Countably infinite subsets.

Enumerating a set implies countable.
Corollary: Any subset $T$ of a countable set $S$ is countable.

Enumerate $T$ as follows:
Get next element, $x$, of $S$,
output only if $x \in T$.

Implications:
$\mathbb{Z}^+$ is countable.
It is infinite since the list goes on.
There is a bijection with the natural numbers.
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$\mathbb{Z}^+$ is countable.
It is infinite since the list goes on.
There is a bijection with the natural numbers.
So it is countably infinite.

All countably infinite sets have the same cardinality.
Enumeration example.

All binary strings.
Enumeration example.

All binary strings. 
\[ B = \{ 0, 1 \}^\ast. \]
Enumeration example.

All binary strings.

\[ B = \{0, 1\}^*. \]

\[ B = \{\phi, \} \]
All binary strings.

\( B = \{0, 1\}^* \).

\( B = \{\phi, 0, \)
All binary strings.

\[ B = \{0, 1\}^*. \]

\[ B = \{\phi, 0, 1, \cdots\} \]

\( \phi \) is empty string.

For any string, it appears at some position in the list.

If \( n \) bits, it will appear before position \( 2^n + 1 \).

Should be careful here.
All binary strings.  
\( B = \{0, 1\}^* \).  
\( B = \{\phi, 0, 1, 00, \ldots\} \).
All binary strings.

\[ B = \{0, 1\}^*. \]

\[ B = \{\emptyset, 0, 1, 00, 01, 10, 11, \ldots\}. \]

\( \emptyset \) is empty string. For any string, it appears at some position in the list. If \( n \) bits, it will appear before position \( 2^n + 1 \). Should be careful here.
All binary strings.

$B = \{0, 1\}^*.$

$B = \{\phi, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots \}.$
Enumeration example.

All binary strings.
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\( B = \{\phi; , 0, 00, 000, 0000, \ldots \} \)
Enumeration example.

All binary strings.
\( B = \{0, 1\}^* \).

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Should be careful here.

\( B = \{\phi, 0, 00, 000, 0000, \ldots\} \)

Never get to 1.
More fractions?

Enumerate the rational numbers in order...
More fractions?

Enumerate the rational numbers in order...
0, ..., 1/2, ..
More fractions?

Enumerate the rational numbers in order...
0, ..., 1/2, ...

Where is 1/2 in list?
More fractions?

Enumerate the rational numbers in order...
0, ..., 1/2, ...
Where is 1/2 in list?
After 1/3, which is after 1/4, which is after 1/5...
More fractions?

Enumerate the rational numbers in order...
0, ..., 1/2, ...

Where is 1/2 in list?
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A thing about fractions:
More fractions?

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Where is 1/2 in list?
After 1/3, which is after 1/4, which is after 1/5...

A thing about fractions:
any two fractions has another fraction between it.
Enumerate the rational numbers in order...
0, ..., 1/2, ...
Where is 1/2 in list?
After 1/3, which is after 1/4, which is after 1/5...
A thing about fractions:
any two fractions has another fraction between it.
Can’t even get to “next” fraction!
More fractions?

Enumerate the rational numbers in order...
0, ... , 1/2, ...

Where is 1/2 in list?

After 1/3, which is after 1/4, which is after 1/5...

A thing about fractions:
any two fractions has another fraction between it.

Can’t even get to “next” fraction!

Can’t list in “order”.

Pairs of natural numbers.

Consider pairs of natural numbers: $N \times N$
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E.g.: (1, 2), (100, 30), etc.
Pairs of natural numbers.

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For finite sets $S_1$ and $S_2$, 
Consider pairs of natural numbers: $N \times N$
E.g.: $(1, 2), (100, 30)$, etc.

For finite sets $S_1$ and $S_2$, then $S_1 \times S_2$
Pairs of natural numbers.

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E.g.: \((1,2), (100,30), \text{etc.}\)

For finite sets \( S_1 \) and \( S_2 \),
then \( S_1 \times S_2 \)
has size \(|S_1| \times |S_2|\).
Consider pairs of natural numbers: \( N \times N \)
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So, $N \times N$ is countably infinite.
Consider pairs of natural numbers: \( N \times N \)

E.g.: \((1, 2), (100, 30), \text{ etc.}\)

For finite sets \( S_1 \) and \( S_2 \),
then \( S_1 \times S_2 \)
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So, \( N \times N \) is countably infinite squared
Pairs of natural numbers.

Consider pairs of natural numbers: \( N \times N \)
E.g.: \((1,2), (100,30), \) etc.

For finite sets \( S_1 \) and \( S_2 \),
then \( S_1 \times S_2 \)
has size \(|S_1| \times |S_2|\).

So, \( N \times N \) is countably infinite squared ???
Pairs of natural numbers.

Enumerate in list:

- (0, 0)
- (1, 0)
- (0, 1)
- (2, 0)
- (1, 1)
- (0, 2)
- ...

The pair \((a, b)\), is in first \((a + b + 1)(a + b)/2\) elements of the list! (i.e., “triangle”).

Countably infinite.

Same size as the natural numbers!!
Pairs of natural numbers.

Enumerate in list:
(0, 0),
(1, 0),
(0, 1),
(2, 0),
(1, 1),
(0, 2),
......
Pairs of natural numbers.

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Pairs of natural numbers.

Enumerate in list:
(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2), ……

The pair (a, b), is in first (a + b + 1)(a + b)/2 elements of list! (i.e., “triangle”).

Countably infinite.
Same size as the natural numbers!!
Pairs of natural numbers.

Enumerate in list:

\[(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2), \ldots \]
Pairs of natural numbers.

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Countably infinite.

Same size as the natural numbers!!
Rationals?

Positive rational number.
Rationals?

Positive rational number.
Lowest terms: \( \frac{a}{b} \)
Positive rational number.
Lowest terms: \( \frac{a}{b} \)
\( a, b \in N \)
Positive rational number.
Lowest terms: \( a/b \)
\( a, b \in \mathbb{N} \)
with \( \gcd(a, b) = 1 \).
Rationals?

Positive rational number.
Lowest terms: \( \frac{a}{b} \)
\( a, b \in N \)
with \( \text{gcd}(a, b) = 1 \).
Infinite subset of \( N \times N \).
Rationals?

Positive rational number.
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Countably infinite!

All rational numbers?
Rationals?

Positive rational number.
Lowest terms: \( a/b \)
\( a, b \in N \)
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Infinite subset of \( N \times N \).

Countably infinite!

All rational numbers?

Negative rationals are countable.
Positive rational number.  
Lowest terms: $a/b$  
$a, b \in N$  
with $gcd(a, b) = 1$.  

Infinite subset of $N \times N$.  

Countably infinite!  

All rational numbers?  

Negative rationals are countable.  (Same size as positive rationals.)
Positive rational number.
Lowest terms: $a/b$

$a, b \in \mathbb{N}$

with $gcd(a, b) = 1$.

Infinite subset of $\mathbb{N} \times \mathbb{N}$.

Countably infinite!

All rational numbers?

Negative rationals are countable. (Same size as positive rationals.)

Put all rational numbers in a list.
Rationals?

Positive rational number.
Lowest terms: \( a/b \)
\( a, b \in N \)
with \( \gcd(a, b) = 1 \).

Infinite subset of \( N \times N \).
Countably infinite!

All rational numbers?
Negative rationals are countable. (Same size as positive rationals.)

Put all rational numbers in a list.
First negative, then nonegative
Positive rational number.
Lowest terms: \( \frac{a}{b} \)
\( a, b \in N \)
with \( \gcd(a, b) = 1 \).

Infinite subset of \( N \times N \).
Countably infinite!

All rational numbers?

Negative rationals are countable. (Same size as positive rationals.)
Put all rational numbers in a list.
First negative, then nonegative ??
Rationals?

Positive rational number.
Lowest terms: \( \frac{a}{b} \)
\( a, b \in N \)
with \( \gcd(a, b) = 1 \).

Infinite subset of \( N \times N \).
Countably infinite!

All rational numbers?

Negative rationals are countable. (Same size as positive rationals.)
Put all rational numbers in a list.
First negative, then nonegative ??? No!
Rationals?

Positive rational number.
Lowest terms: $a/b$
$a, b \in N$
with $gcd(a, b) = 1$.

Infinite subset of $N \times N$.

Countably infinite!

All rational numbers?
Negative rationals are countable. (Same size as positive rationals.)

Put all rational numbers in a list.
First negative, then nonegative ??? No!
Repeatedly and alternatively take one from each list.
Rationals?

Positive rational number.
Lowest terms: \( a/b \)
\( a, b \in N \)
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Infinite subset of \( N \times N \).

Countably infinite!

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Interleave Streams in 61A
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Lowest terms: \(a/b\)
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Infinite subset of \(N \times N\).

Countably infinite!

All rational numbers?

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Put all rational numbers in a list.

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Interleave Streams in 61A

The rationals are countably infinite.
Real numbers.

Real numbers are same size as integers?
The reals.

Are the set of reals countable?
Are the set of reals countable?
Lets consider the reals $[0, 1]$. 
Are the set of reals countable?

Let's consider the reals [0, 1].

Each real has a decimal representation.
Are the set of reals countable?

Let's consider the reals $[0, 1]$.

Each real has a decimal representation.

$0.500000000...$
Are the set of reals countable?

Let's consider the reals $[0, 1]$.

Each real has a decimal representation.

$.500000000...$ (1/2)
The reals.

Are the set of reals countable?

Lets consider the reals \([0, 1]\).

Each real has a decimal representation.

\(0.500000000\ldots\) (1/2)

\(0.785398162\ldots\)
Are the set of reals countable?

Let's consider the reals \([0, 1]\).

Each real has a decimal representation.
.500000000... \((1/2)\)
.785398162... \(\pi/4\)
The reals.

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Let's consider the reals \([0, 1]\).

Each real has a decimal representation.

- .500000000... \((1/2)\)
- .785398162... \(\pi/4\)
- .367879441...
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.500000000... (1/2)

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.367879441... 1/e
Are the set of reals countable?

Let's consider the reals $[0, 1]$.

Each real has a decimal representation.

.500000000... (1/2)
.785398162... $\pi/4$
.367879441... $1/e$
.632120558...
Are the set of reals countable?

Let's consider the reals $[0, 1]$.

Each real has a decimal representation.

- $0.500000000...$ (1/2)
- $0.785398162...$ $\pi/4$
- $0.367879441...$ $1/e$
- $0.632120558...$ $1 - 1/e$
The reals.

Are the set of reals countable?

Let's consider the reals $[0, 1]$.

Each real has a decimal representation.

.500000000... (1/2)
.785398162... π/4
.367879441... 1/e
.632120558... 1 − 1/e
.345212312...
Are the set of reals countable?

Let's consider the reals $[0, 1]$.

Each real has a decimal representation.

.500000000... (1/2)
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.345212312... Some real number
Are the set of reals countable?

Let's consider the reals \([0, 1]\).

Each real has a decimal representation.
\[.500000000... \ (1/2)\]
\[.785398162... \ \pi/4\]
\[.367879441... \ 1/e\]
\[.632120558... \ 1 - 1/e\]
\[.345212312... \ Some \ real \ number\]
Diagonalization.

If countable, there a listing, \( L \) contains all reals.
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example 0: .500000000...
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

$0$: \(0.500000000...\)

$1$: \(0.785398162...\)

Construct "diagonal" number:

\[
0.776777...\]

Diagonal Number:

Digit $i$ is 7 if number $i$'s $i$th digit is not 7 and 6 otherwise.

Diagonal number for a list differs from every number in list!

Diagonal number not in list.

Diagonal number is real.

Contradiction!

Subset $[0, 1]$ is not countable!!
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...

Construct “diagonal” number: .77677...

Diagonal Number:

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0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .5000000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: \[.500000000...\]
1: \[.785398162...\]
2: \[.367879441...\]
3: \[.632120558...\]
4: \[.345212312...\]
5: \[.456789012...\]
...
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

Construct “diagonal” number:
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .5000000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
...

Construct “diagonal” number: .7
Diagonalization.

If countable, there a listing, \( L \) contains all reals. For example

0: \( .500000000... \)
1: \( .785398162... \)
2: \( .367879441... \)
3: \( .632120558... \)
4: \( .345212312... \)

Construct “diagonal” number: \( .77 \)
If countable, there a listing, \( L \) contains all reals. For example

0: \( .5000000000... \)
1: \( .785398162... \)
2: \( .367879441... \)
3: \( .632120558... \)
4: \( .345212312... \)

Construct “diagonal” number: \( .776 \)
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: \(0.500000000\ldots\)
1: \(0.785398162\ldots\)
2: \(0.367879441\ldots\)
3: \(0.632120558\ldots\)
4: \(0.345212312\ldots\)

Construct “diagonal” number: \(0.7767\ldots\)
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: \textcolor{red}{.5}000000000...
1: \textcolor{red}{.7}85398162...
2: \textcolor{red}{.3}67879441...
3: \textcolor{red}{.6}32120558...
4: \textcolor{red}{.3}45212312...

Construct “diagonal” number: \textcolor{red}{.77677}
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

Construct “diagonal” number: .77677...
Diagonalization.

If countable, there a listing, \( L \) contains all reals. For example

0: .5000000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

: 

Construct “diagonal” number: .77677…

Diagonal Number:
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: $0.500000000...$
1: $0.785398162...$
2: $0.367879441...$
3: $0.632120558...$
4: $0.345212312...$

Construct “diagonal” number: $0.77677...$

Diagonal Number: Digit $i$ is 7 if number $i$’s $i$th digit is not 7
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

Construct “diagonal” number: .77677...

Diagonal Number: Digit $i$ is 7 if number $i$’s $i$th digit is not 7 and 6 otherwise.
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

Construct “diagonal” number: .77677...

Diagonal Number: Digit $i$ is 7 if number $i$’s $i$th digit is not 7 and 6 otherwise.

Diagonal number for a list differs from every number in list!
Diagonalization.

If countable, there a listing, \( L \) contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

::

Construct “diagonal” number: .77677…

Diagonal Number: Digit \( i \) is 7 if number \( i \)’s \( i \)th digit is not 7 and 6 otherwise.

Diagonal number for a list differs from every number in list! Diagonal number not in list.
Diagonalization.

If countable, there a listing, \( L \) contains all reals. For example

0: \( .5000000000... \)
1: \( .785398162... \)
2: \( .367879441... \)
3: \( .632120558... \)
4: \( .345212312... \)

: 

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Contradiction!

Subset $[0, 1]$ is not countable!!
All reals?

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What about all reals?
All reals?

Subset $[0, 1]$ is not countable!!
What about all reals?
No.
All reals?

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What about all reals?
No.

Any subset of a countable set is countable.
All reals?

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What about all reals?
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If reals are countable then so is [0, 1].
Diagonalization.

1. Assume that a set $S$ can be enumerated.
Diagonalization.

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Contradiction.
Diagonalization.

1. Assume that a set $S$ can be enumerated.
2. Consider an arbitrary list of all the elements of $S$.
3. Use the diagonal from the list to construct a new element $t$.
4. Show that $t$ is different from all elements in the list $\Rightarrow t$ is not in the list.
5. Show that $t$ is in $S$.
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Another diagonalization.

The set of all subsets of \( N \).
Another diagonalization.

The set of all subsets of $N$.

Example subsets of $N$: \{0\},
Another diagonalization.

The set of all subsets of \( N \).

Example subsets of \( N \): \( \{0\} \), \( \{0, \ldots, 7\} \),
Another diagonalization.

The set of all subsets of $\mathbb{N}$.

Example subsets of $\mathbb{N}$: \{0\}, \{0, \ldots, 7\},
Another diagonalization.

The set of all subsets of $N$.

Example subsets of $N$: $\{0\}$, $\{0, \ldots, 7\}$, evens,
Another diagonalization.

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Example subsets of $\mathbb{N}$: $\{0\}$, $\{0, \ldots, 7\}$, evens, odds,
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Example subsets of $N$: $\{0\}$, $\{0, \ldots, 7\}$, evens, odds, primes,
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Example subsets of \( N \): \{0\}, \{0,\ldots,7\}, evens, odds, primes,

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There is a listing, \( L \), that contains all subsets of \( N \).
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Define a diagonal set, \( D \):
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There is a listing, $L$, that contains all subsets of $N$.

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If $i$th set in $L$ does not contain $i$, $i \in D$. 
Another diagonalization.

The set of all subsets of \( \mathbb{N} \).

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\( L \) does not contain all subsets of \( \mathbb{N} \).
Contradiction.

Theorem: The set of all subsets of \( \mathbb{N} \) is not countable.

(The set of all subsets of \( S \), is the powerset of \( \mathbb{N} \).)
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Contradiction.

**Theorem:** The set of all subsets of $N$ is not countable. (The set of all subsets of $S$, is the **powerset** of $N$.)
Diagonalize Natural Number.

Natural numbers have a listing, \( L \).
Natural numbers have a listing, $L$.

Make a diagonal number, $D$: differ from $i$th element of $L$ in $i$th digit.
Natural numbers have a listing, $L$.

Make a diagonal number, $D$:

differ from $i$th element of $L$ in $i$th digit.

Differs from all elements of listing.
Diagonalize Natural Number.

Natural numbers have a listing, $L$.

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“Construction” requires an infinite number of digits.
The Continuum hypothesis.

There is no set with cardinality between the naturals and the reals.
The Continuum hypothesis.

There is no set with cardinality between the naturals and the reals. First of Hilbert’s problems!
Cardinalities of uncountable sets?

Cardinality of $[0, 1]$ smaller than all the reals?
Cardinalities of uncountable sets?

Cardinality of $[0, 1]$ smaller than all the reals?

$f : R^+ \to [0, 1]$. 
Cardinalities of uncountable sets?

Cardinality of $[0, 1]$ smaller than all the reals?

$f : R^+ \rightarrow [0, 1].$

$$f(x) = \begin{cases} 
  x + \frac{1}{2} & \text{if } 0 \leq x \leq \frac{1}{2} \\
  \frac{1}{4x} & \text{if } x > \frac{1}{2}
\end{cases}$$
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One to one.
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If one is in $[0, 1/2]$ and one isn’t,
Cardinalities of uncountable sets?

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Bijection!
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If both in $[0, 1/2]$, a shift $\iff f(x) \neq f(y)$.

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If one is in $[0, 1/2]$ and one isn’t, different ranges $\iff f(x) \neq f(y)$.

Bijection!

$[0, 1]$ is same cardinality as nonnegative reals!
Generalized Continuum hypothesis.

There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.
Generalized Continuum hypothesis.

There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.

The powerset of a set is the set of all subsets.
Resolution of hypothesis?

Gödel. 1940.

Can't use math!

If math doesn't contain a contradiction.

This statement is a lie.

Is the statement above true?

The barber shaves every person who does not shave themselves.

Who shaves the barber?

Self reference.
Resolution of hypothesis?

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More on...

...Tuesday..
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