Today.

More Counting.
Today.

More Counting.
Probability.
Sampling and counting.

**First rule:** \( n_1 \times n_2 \cdots \times n_3. \)
Sampling and counting.

First rule: \( n_1 \times n_2 \times \cdots \times n_3 \).

\( k \) Samples with replacement from \( n \) items: \( n^k \).
Sampling and counting.

**First rule:** \( n_1 \times n_2 \cdots \times n_3 \).

\( k \) Samples with replacement from \( n \) items: \( n^k \).
Sample without replacement: \( \frac{n!}{(n-k)!} \).
Sampling and counting.

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\( k \) Samples with replacement from \( n \) items: \( n^k \).
Sample without replacement: \( \frac{n!}{(n-k)!} \).

**Second rule:** when order doesn’t matter divide..when possible.
Sampling and counting.

**First rule:** \( n_1 \times n_2 \cdots \times n_3 \).

\( k \) Samples with replacement from \( n \) items: \( n^k \).

Sample without replacement: \( \frac{n!}{(n-k)!} \).

**Second rule:** when order doesn’t matter divide..when possible.

Sample without replacement and order doesn’t matter: \( \binom{n}{k} = \frac{n!}{(n-k)!k!} \).

“\( n \) choose \( k \)”
Sampling and counting.

**First rule:** \( n_1 \times n_2 \cdots \times n_3 \).

\( k \) Samples with replacement from \( n \) items: \( n^k \).
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Sample without replacement and order doesn’t matter: \( \binom{n}{k} = \frac{n!}{(n-k)!k!} \).
“\( n \) choose \( k \)”

**One-to-one rule:** equal in number if one-to-one correspondence.
Sampling and counting.

First rule: \( n_1 \times n_2 \cdots \times n_3 \).

\( k \) Samples with replacement from \( n \) items: \( n^k \).

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Second rule: when order doesn’t matter divide...when possible.

Sample without replacement and order doesn’t matter: \( \binom{n}{k} = \frac{n!}{(n-k)!k!} \).

“\( n \) choose \( k \)”

One-to-one rule: equal in number if one-to-one correspondence.

Sample with replacement and order doesn’t matter: \( \binom{k+n-1}{n-1} \).
Balls in bins.

“$k$ Balls in $n$ bins” $\equiv$ “$k$ samples from $n$ possibilities.”
Balls in bins.

“$k$ Balls in $n$ bins” $\equiv$ “$k$ samples from $n$ possibilities.”
“indistinguishable balls” $\equiv$ “order doesn’t matter”
Balls in bins.

“$k$ Balls in $n$ bins” $\equiv$ “$k$ samples from $n$ possibilities.”

“indistinguishable balls” $\equiv$ “order doesn’t matter”

“only one ball in each bin” $\equiv$ “without replacement”
Balls in bins.

“k Balls in n bins” ≡ “k samples from n possibilities.”
“indistinguishable balls” ≡ “order doesn’t matter”
“only one ball in each bin” ≡ “without replacement”
5 balls into 10 bins
Balls in bins.

“$k$ Balls in $n$ bins” $\equiv$ “$k$ samples from $n$ possibilities.”

“indistinguishable balls” $\equiv$ “order doesn’t matter”

“only one ball in each bin” $\equiv$ “without replacement”

5 balls into 10 bins
5 samples from 10 possibilities with replacement
“k Balls in n bins” ≡ “k samples from n possibilities.”

“indistinguishable balls” ≡ “order doesn’t matter”

“only one ball in each bin” ≡ “without replacement”

5 balls into 10 bins
5 samples from 10 possibilities with replacement

Example: 5 digit phone numbers.
5 Balls/places choose from 10 bins/digits.
Balls in bins.

“$k$ Balls in $n$ bins” $\equiv$ “$k$ samples from $n$ possibilities.”
“indistinguishable balls” $\equiv$ “order doesn’t matter”
“only one ball in each bin” $\equiv$ “without replacement”

5 balls into 10 bins
5 samples from 10 possibilities with replacement
  Example: 5 digit phone numbers.
  5 Balls/places choose from 10 bins/digits.

5 indistinguishable balls into 52 bins only one ball in each bin
Balls in bins.

“k Balls in n bins” ≡ “k samples from n possibilities.”

“indistinguishable balls” ≡ “order doesn’t matter”

“only one ball in each bin” ≡ “without replacement”

5 balls into 10 bins
5 samples from 10 possibilities with replacement
  Example: 5 digit phone numbers.
  5 Balls/places choose from 10 bins/digits.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement
Balls in bins.

“\(k\) Balls in \(n\) bins” \(\equiv\) “\(k\) samples from \(n\) possibilities.”

“indistinguishable balls” \(\equiv\) “order doesn’t matter”

“only one ball in each bin” \(\equiv\) “without replacement”

5 balls into 10 bins
5 samples from 10 possibilities with replacement
   Example: 5 digit phone numbers.
   5 Balls/places choose from 10 bins/digits.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement
   Example: Poker hands.
   5 balls/cards into 52 bins/possible cards.
Balls in bins.

“$k$ Balls in $n$ bins” ≡ “$k$ samples from $n$ possibilities.”
“indistinguishable balls” ≡ “order doesn’t matter”
“only one ball in each bin” ≡ “without replacement”

5 balls into 10 bins
5 samples from 10 possibilities with replacement
   Example: 5 digit phone numbers.
   5 Balls/places choose from 10 bins/digits.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement
   Example: Poker hands.
   5 balls/cards into 52 bins/possible cards.

5 indistinguishable balls into 3 bins
“k Balls in n bins” ≡ “k samples from n possibilities.”

“indistinguishable balls” ≡ “order doesn’t matter”

“only one ball in each bin” ≡ “without replacement”

5 balls into 10 bins
5 samples from 10 possibilities with replacement
   Example: 5 digit phone numbers.
   5 Balls/places choose from 10 bins/digits.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement
   Example: Poker hands.
   5 balls/cards into 52 bins/possible cards.

5 indistinguishable balls into 3 bins
5 samples from 3 possibilities with replacement and no order
Balls in bins.

“$k$ Balls in $n$ bins” ≡ “$k$ samples from $n$ possibilities.”

“indistinguishable balls” ≡ “order doesn’t matter”

“only one ball in each bin” ≡ “without replacement”

5 balls into 10 bins
5 samples from 10 possibilities with replacement
  Example: 5 digit phone numbers.
  5 Balls/places choose from 10 bins/digits.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement
  Example: Poker hands.
  5 balls/cards into 52 bins/possible cards.

5 indistinguishable balls into 3 bins
5 samples from 3 possibilities with replacement and no order
  Dividing 5 dollars among Alice, Bob and Eve.
  5 dollars/balls choose from 3 people/bins.
Sum Rule

Two indistinguishable jokers in 54 card deck.
How many 5 card poker hands?
Sum Rule

Two indistinguishable jokers in 54 card deck. How many 5 card poker hands?

Sum rule: Can sum over disjoint sets.
Sum Rule

Two indistinguishable jokers in 54 card deck.
How many 5 card poker hands?
Sum rule: Can sum over disjoint sets.

“exclusive” or Two Jokers

No jokers

“exclusive” or One Joker

(52 \choose 5) + (52 \choose 4) + (52 \choose 3).

Two distinguishable jokers in 54 card deck.
How many 5 card poker hands?

Choose 4 cards plus one of 2 jokers!

(52 \choose 5) + 2 \times (52 \choose 4) + (52 \choose 3).

Wait a minute!

Same as choosing 5 cards from 54

or (54 \choose 5).

Theorem:

(54 \choose 5) = (52 \choose 5) + 2 \times (52 \choose 4) + (52 \choose 3).

Algebraic Proof:

No need! Above is combinatorial proof.
Two indistinguishable jokers in 54 card deck.
How many 5 card poker hands?

**Sum rule:** Can sum over disjoint sets.

No jokers \( \binom{52}{5} \)

“exclusive” or Two Jokers

\[
\binom{52}{4} + 2 \binom{52}{3}
\]
Sum Rule

Two indistinguishable jokers in 54 card deck. How many 5 card poker hands?

*Sum rule:* Can sum over disjoint sets.

No jokers “exclusive” or One Joker “exclusive” or Two Jokers

\[
\binom{52}{5} + \binom{52}{4} + \binom{52}{3}
\]
Sum Rule

Two indistinguishable jokers in 54 card deck. How many 5 card poker hands?

Sum rule: Can sum over disjoint sets.

No jokers "exclusive" or One Joker "exclusive" or Two Jokers

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\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.
\]
Two indistinguishable jokers in 54 card deck. How many 5 card poker hands?

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\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.
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Two distinguishable jokers in 54 card deck.
Sum Rule

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Two distinguishable jokers in 54 card deck. How many 5 card poker hands?
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How many 5 card poker hands?

**Sum rule: Can sum over disjoint sets.**

No jokers “exclusive” or One Joker “exclusive” or Two Jokers

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\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.
\]

Two distinguishable jokers in 54 card deck.
How many 5 card poker hands?

\[
\binom{52}{5} +
\]

Wait a minute!

Same as choosing 5 cards from 54 or

\[
\binom{54}{5} = \binom{52}{5} + 2 \times \binom{52}{4} + \binom{52}{3}.
\]

Theorem:

\[
\binom{54}{5} = \binom{52}{5} + 2 \times \binom{52}{4} + \binom{52}{3}.
\]

Algebraic Proof:

No need! Above is combinatorial proof.
Sum Rule

Two indistinguishable jokers in 54 card deck. How many 5 card poker hands?

**Sum rule:** Can sum over disjoint sets.

No jokers "exclusive" or One Joker "exclusive" or Two Jokers

\[
\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.
\]

Two distinguishable jokers in 54 card deck. How many 5 card poker hands? Choose 4 cards plus one of 2 jokers!

\[
\binom{52}{5} + 2 \times \binom{52}{4} + \binom{52}{3}.
\]

Wait a minute! Same as choosing 5 cards from 54 or \(\binom{54}{5}\).
Sum Rule

Two indistinguishable jokers in 54 card deck.
How many 5 card poker hands?
**Sum rule: Can sum over disjoint sets.**
No jokers “exclusive” or One Joker “exclusive” or Two Jokers

\[
\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.
\]

Two distinguishable jokers in 54 card deck.
How many 5 card poker hands? **Choose 4 cards plus one of 2 jokers!**

\[
\binom{52}{5} + 2 \times \binom{52}{4} + \binom{52}{3}.
\]
Sum Rule

Two indistinguishable jokers in 54 card deck. How many 5 card poker hands?

**Sum rule: Can sum over disjoint sets.**

No jokers “exclusive” or One Joker “exclusive” or Two Jokers

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\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.
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Two distinguishable jokers in 54 card deck. How many 5 card poker hands? **Choose 4 cards plus one of 2 jokers!**

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\binom{52}{5} + 2 \times \binom{52}{4} + \binom{52}{3}
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Wait a minute!
Sum Rule

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\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.
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Two distinguishable jokers in 54 card deck. How many 5 card poker hands? Choose 4 cards plus one of 2 jokers!

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Wait a minute! Same as choosing 5 cards from 54
Sum Rule

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Two distinguishable jokers in 54 card deck.
How many 5 card poker hands? Choose 4 cards plus one of 2 jokers!

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\binom{52}{5} + 2 \times \binom{52}{4} + \binom{52}{3}
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Wait a minute! Same as choosing 5 cards from 54 or
Two indistinguishable jokers in 54 card deck. How many 5 card poker hands?

**Sum rule: Can sum over disjoint sets.**

No jokers “exclusive” or One Joker “exclusive” or Two Jokers

\[ \binom{52}{5} + \binom{52}{4} + \binom{52}{3}. \]

Two distinguishable jokers in 54 card deck. How many 5 card poker hands? Choose 4 cards plus one of 2 jokers!

\[ \binom{52}{5} + 2 \times \binom{52}{4} + \binom{52}{3} \]

Wait a minute! Same as choosing 5 cards from 54 or

\[ \binom{54}{5} \]

**Theorem:** \( \binom{54}{5} \)
Sum Rule

Two indistinguishable jokers in 54 card deck. How many 5 card poker hands?

**Sum rule:** Can sum over disjoint sets.

No jokers “exclusive” or One Joker “exclusive” or Two Jokers

\[
\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.
\]

Two distinguishable jokers in 54 card deck. How many 5 card poker hands? Choose 4 cards plus one of 2 jokers!

\[
\binom{52}{5} + 2*\binom{52}{4} + \binom{52}{3}
\]

Wait a minute! Same as choosing 5 cards from 54 or

\[
\binom{54}{5}
\]

**Theorem:** \( \binom{54}{5} = \binom{52}{5} + 2*\binom{52}{4} + \binom{52}{3} \).
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\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.
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Two distinguishable jokers in 54 card deck. How many 5 card poker hands? **Choose 4 cards plus one of 2 jokers!**

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\binom{52}{5} + 2 \times \binom{52}{4} + \binom{52}{3}
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Wait a minute! Same as choosing 5 cards from 54 or

\[
\binom{54}{5}
\]

**Theorem:** \( \binom{54}{5} = \binom{52}{5} + 2 \times \binom{52}{4} + \binom{52}{3} \).

**Algebraic Proof:**
Sum Rule

Two indistinguishable jokers in 54 card deck. How many 5 card poker hands?

**Sum rule:** Can sum over disjoint sets.

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\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.
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\binom{52}{5} + 2 \times \binom{52}{4} + \binom{52}{3}
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Wait a minute! Same as choosing 5 cards from 54 or

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\binom{54}{5}
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**Theorem:** \( \binom{54}{5} = \binom{52}{5} + 2 \times \binom{52}{4} + \binom{52}{3} \).

**Algebraic Proof:** No need! Above is combinatorial proof.
Sum Rule

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How many 5 card poker hands?

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How many 5 card poker hands? Choose 4 cards plus one of 2 jokers!

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\binom{52}{5} + 2 \times \binom{52}{4} + \binom{52}{3}
\]

Wait a minute! Same as choosing 5 cards from 54 or

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\binom{54}{5}
\]

**Theorem:** \(\binom{54}{5} = \binom{52}{5} + 2 \times \binom{52}{4} + \binom{52}{3}\).

**Algebraic Proof:** No need! Above is combinatorial proof.
Combinatorial Proofs.

Theorem: \( \binom{n}{k} = \binom{n}{n-k} \)
Combinatorial Proofs.

**Theorem:** \( \binom{n}{k} = \binom{n}{n-k} \)

**Proof:** How many subsets of size \( k \)?
Theorem: \( \binom{n}{k} = \binom{n}{n-k} \)

Proof: How many subsets of size \( k \)? \( \binom{n}{k} \)
Combinatorial Proofs.

**Theorem:** \( \binom{n}{k} = \binom{n}{n-k} \)

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Theorem: \( \binom{n}{k} = \binom{n}{n-k} \)

Proof: How many subsets of size \( k \)? \( \binom{n}{k} \)

How many subsets of size \( k \)?
Choose a subset of size \( n-k \)
Theorem: \( \binom{n}{k} = \binom{n}{n-k} \)

Proof: How many subsets of size \( k \)? \( \binom{n}{k} \)

How many subsets of size \( k \)?
Choose a subset of size \( n-k \) and what’s left out
Theorem: \( \binom{n}{k} = \binom{n}{n-k} \)

Proof: How many subsets of size \( k \)? \( \binom{n}{k} \)

How many subsets of size \( k \)?
Choose a subset of size \( n - k \)
    and what’s left out is a subset of size \( k \).
Theorem: \( \binom{n}{k} = \binom{n}{n-k} \)

Proof: How many subsets of size \( k \)? \( \binom{n}{k} \)

How many subsets of size \( k \)?
Choose a subset of size \( n - k \)
and what’s left out is a subset of size \( k \).
Choosing a subset of size \( k \) is same
Combinatorial Proofs.

**Theorem:** \( \binom{n}{k} = \binom{n}{n-k} \)

**Proof:** How many subsets of size \( k \)? \( \binom{n}{k} \)

How many subsets of size \( k \)?
Choose a subset of size \( n-k \)
and what’s left out is a subset of size \( k \).
Choosing a subset of size \( k \) is same
as choosing \( n-k \) elements to not take.
Combinatorial Proofs.

Theorem: \( \binom{n}{k} = \binom{n}{n-k} \)

Proof: How many subsets of size \( k \)? \( \binom{n}{k} \)

How many subsets of size \( k \)?
Choose a subset of size \( n-k \) and what’s left out is a subset of size \( k \).
Choosing a subset of size \( k \) is same as choosing \( n-k \) elements to not take.
\( \Rightarrow \binom{n}{n-k} \) subsets of size \( k \).
Theorem: \( \binom{n}{k} = \binom{n}{n-k} \)

Proof: How many subsets of size \( k \)? \( \binom{n}{k} \)

How many subsets of size \( k \)?
Choose a subset of size \( n - k \)
and what’s left out is a subset of size \( k \).
Choosing a subset of size \( k \) is same
as choosing \( n - k \) elements to not take.
\( \implies \binom{n}{n-k} \) subsets of size \( k \).
Pascal’s Triangle

Row $n$: coefficients of $(1 + x)^n = (1 + x)(1 + x) \cdots (1 + x)$.

- Foil (4 terms): $2^n$ terms: choose 1 or $x$ from each factor of $(1 + x)$.
- Simplify: collect all terms corresponding to $x^k$.
- Coefficient of $x^k$ is $\binom{n}{k}$: choose $k$ factors where $x$ is in product.

\[
\begin{array}{ccccccc}
& & & & 0 & & 0 \\
& & & 1 & 1 & & \\
& & 1 & 2 & 1 & & \\
& 1 & 3 & 3 & 1 & & \\
1 & 4 & 6 & 4 & 1 & & \\
\end{array}
\]

Pascal’s rule $\Rightarrow \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$. 
Pascal’s Triangle

Row \( n \): coefficients of \((1 + x)^n\) = \((1 + x)(1 + x) \cdots (1 + x)\).

Foil (4 terms): 2 terms: choose 1 or \(x\) from each factor of \((1 + x)\).

Simplify: collect all terms corresponding to \(x^k\).

Coefficient of \(x^k\) is \(\binom{n}{k}\): choose \(k\) factors where \(x\) is in product.

\[
\begin{array}{cccc}
0 \\
1 & 1 \\
2 & 1 & 1 & 1 \\
3 & 1 & 2 & 1 & 1 \\
4 & 1 & 3 & 3 & 1 & 1 \\
5 & 1 & 4 & 6 & 4 & 1 & 1 \\
\end{array}
\]

Pascal’s rule: \(\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}\).
Pascal’s Triangle

\[
\begin{array}{cccc}
0 \\
1 & 1 \\
1 & 2 & 1 \\
\end{array}
\]

Row \( n \): coefficients of \((1 + x)^n\) = \((1 + x)(1 + x)\cdots(1 + x)\).

Foil (4 terms): \(2^n\) terms: choose 1 or \(x\) from each factor of \((1 + x)\).

Simplify: collect all terms corresponding to \(x^k\).

Coefficient of \(x^k\) is \(\binom{n}{k}\): choose \(k\) factors where \(x\) is in product.

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
1 & 2 & 1 & 1 \\
\end{array}
\]

Pascal’s rule = \(\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}\).
Pascal’s Triangle

\[
\begin{array}{c}
0 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1
\end{array}
\]
Pascal’s Triangle

\[
\begin{array}{cccccc}
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
\end{array}
\]

Row \( n \): coefficients of \((1 + x)^n\) = \((1 + x)(1 + x) \cdots (1 + x)\).

Foil (4 terms) on steroids: \(2^n\) terms: choose 1 or \(x\) from each factor of \((1 + x)\).

Simplify: collect all terms corresponding to \(x^k\).

Coefficient of \(x^k\) is \(\binom{n}{k}\): choose \(k\) factors where \(x\) is in product.

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
\end{array}
\]

Pascal’s rule \(\Rightarrow\) \(\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}\).
Pascal’s Triangle

0
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Row \( n \): coefficients of \((1 + x)^n\) = \((1 + x)(1 + x)(1 + x) \cdots\).

Foil (4 terms) on steroids: \(2^n\) terms: choose 1 or \(x\) from each factor of \((1 + x)\).

Simplify: collect all terms corresponding to \(x^k\).

Coefficient of \(x^k\) is \( \binom{n}{k} \): choose \(k\) factors where \(x\) is in product.

\[
\begin{array}{cccc}
0 & 1 & 1 & 1 & 1 \\
1 & 2 & 1 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
\end{array}
\]
Row $n$: coefficients of $(1 + x)^n = (1 + x)(1 + x) \cdots (1 + x)$. 

Pascal’s Triangle
Pascal’s Triangle

Row \( n \): coefficients of \((1 + x)^n = (1 + x)(1 + x) \cdots (1 + x)\).

Foil (4 terms)
Pascal’s Triangle

Row $n$: coefficients of $(1 + x)^n = (1 + x)(1 + x) \cdots (1 + x)$.

Foil (4 terms) on steroids:
Pascal’s Triangle

Row $n$: coefficients of $(1 + x)^n = (1 + x)(1 + x) \cdots (1 + x)$.

Foil (4 terms) on steroids:
$2^n$ terms:
Pascal’s Triangle

Row $n$: coefficients of $(1 + x)^n = (1 + x)(1 + x) \cdots (1 + x)$.

Foil (4 terms) on steroids:
$2^n$ terms: choose 1 or $x$ from each factor of $(1 + x)$. 

```plaintext
0
1 1
1 2 1
1 3 3 1
1 4 6 4 1
```
Pascal’s Triangle

\[
\begin{array}{ccccccc}
0 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
\end{array}
\]

Row \( n \): coefficients of \((1 + x)^n = (1 + x)(1 + x) \cdots (1 + x)\).

Foil (4 terms) on steroids:
\(2^n\) terms: choose 1 or \(x\) from each factor of \((1 + x)\).
Pascal’s Triangle

Row $n$: coefficients of $(1 + x)^n = (1 + x)(1 + x) \cdots (1 + x)$.

Foil (4 terms) on steroids:

- $2^n$ terms: choose 1 or $x$ from each factor of $(1 + x)$.

Simplify: collect all terms corresponding to $x^k$. 

<p>| | | | | | |</p>
<table>
<thead>
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</thead>
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<tr>
<td>0</td>
<td>1</td>
<td>1</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
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<td>1</td>
<td>3</td>
<td>3</td>
<td>1</td>
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</tr>
<tr>
<td>1</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Pascal’s rule $\Rightarrow (n + 1)^k = (n)^k + (n - 1)^k$. 

Coefficient of $x^k$ is $(n \choose k)$: choose $k$ factors where $x$ is in product.

\[ (0 \ 0) \ (1 \ 0) \ (1 \ 1) \ (2 \ 0) \ (2 \ 1) \ (2 \ 2) \ (3 \ 0) \ (3 \ 1) \ (3 \ 2) \ (3 \ 3) \]
Pascal’s Triangle

\[
\begin{array}{cccccc}
0 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
\end{array}
\]

Row \( n \): coefficients of \((1 + x)^n = (1 + x)(1 + x) \cdots (1 + x)\).

Foil (4 terms) on steroids:
\(2^n\) terms: choose 1 or \(x\) from each factor of \((1 + x)\).

Simplify: collect all terms corresponding to \(x^k\).
Coefficient of \(x^k\) is \( \binom{n}{k} \): choose \(k\) factors where \(x\) is in product.
Pascal’s Triangle

\[
\begin{array}{cccccc}
0 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
\end{array}
\]

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\[
\begin{array}{cccc}
(0) \\
(1) \\
(0) & (1) \\
\end{array}
\]

Pascal’s rule

\[
\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}
\]
Pascal’s Triangle

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- Coefficient of $x^k$ is $\binom{n}{k}$: choose $k$ factors where $x$ is in product.

\[
\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}
\end{array}
\]
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Coefficient of $x^k$ is $\binom{n}{k}$: choose $k$ factors where $x$ is in product.

Pascal’s rule $\implies \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$. 
Combinatorial Proofs.

Theorem: \(\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}\).

Proof: How many size \(k\) subsets of \(n+1\)?
Combinatorial Proofs.

**Theorem:** \( \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \).

**Proof:** How many size \( k \) subsets of \( n+1 \)? \( \binom{n+1}{k} \).

How many contain the first element?
Choose first element, need to choose \( k-1 \) more from remaining \( n \) elements.

\[ \Rightarrow \binom{n}{k-1} \]

How many don't contain the first element?
Need to choose \( k \) elements from remaining \( n \) elts.

\[ \Rightarrow \binom{n}{k} \]

So,

\[ \binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k} \]
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**Theorem:** \( \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \).

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Theorem: \( \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \).

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\[ \Rightarrow \binom{n}{k} \]
Combinatorial Proofs.

**Theorem:** \( {n+1 \choose k} = {n \choose k} + {n \choose k-1} \).

**Proof:** How many size \( k \) subsets of \( n+1 \)? \( {n+1 \choose k} \).

How many size \( k \) subsets of \( n+1 \)?
How many contain the first element?
Choose first element,
need to choose \( k - 1 \) more from remaining \( n \) elements.
\[ \Rightarrow {n \choose k-1} \]

How many don’t contain the first element?
Need to choose \( k \) elements from remaining \( n \) elts.
\[ \Rightarrow {n \choose k} \]

So, \( {n \choose k-1} + {n \choose k} \).
Theorem: \( \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \).

Proof: How many size \( k \) subsets of \( n + 1 \)? \( \binom{n+1}{k} \).

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How many contain the first element?
Choose first element, need to choose \( k - 1 \) more from remaining \( n \) elements.
\[ \implies \binom{n}{k-1} \]

How many don’t contain the first element?
Need to choose \( k \) elements from remaining \( n \) elts.
\[ \implies \binom{n}{k} \]

So, \( \binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k} \). \qed
Combinatorial Proof.

**Theorem:** \( \binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1} \).
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Proof: Consider size \( k \) subset where \( i \) is the first element chosen.
Combinatorial Proof.

**Theorem:** \( \binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1} \).

**Proof:** Consider size \( k \) subset where \( i \) is the first element chosen. 

\[ \{1, \ldots, i, \ldots, n\} \]

Must choose \( k - 1 \) elements from \( n - i \) remaining elements.
Combinatorial Proof.

**Theorem:** \( \binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}. \)

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\{1, \ldots, i, \ldots, n\}
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\( \Rightarrow \binom{n-i}{k-1} \) such subsets.

Note term \( \binom{n-i}{k-1} \) corresponds to subsets where first item is \( i \).
Combinatorial Proof.

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\[ \Rightarrow \binom{n-i}{k-1} \text{ such subsets.} \]

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Do the terms correspond to disjoint Groups?
Combinatorial Proof.

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Do the terms correspond to disjoint Groups? Yes?
Combinatorial Proof.

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\Rightarrow \binom{n-i}{k-1} \text{ such subsets.}
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Do the terms correspond to disjoint Groups? Yes? No?
**Combinatorial Proof.**

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Do the terms correspond to disjoint Groups? Yes? No?

Any pair of subsets in different Groups have different first items.
Combinatorial Proof.

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So Yes!!
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Do the terms correspond to disjoint Groups? Yes? No?
   Any pair of subsets in different Groups have different first items.
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Add their sizes up to get the total number of subsets of size \( k \).
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Do the terms correspond to disjoint Groups? Yes? No?

Any pair of subsets in different Groups have different first items.

So Yes!!

Add their sizes up to get the total number of subsets of size \( k \) which is also \( \binom{n}{k} \).
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**Theorem:** \( \binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1} \).

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Binomial Theorem: \( x = 1 \)

**Theorem:** \( 2^n = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{0} \)
Binomial Theorem: $x = 1$

**Theorem:** $2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$

**Proof:** How many subsets of \{1, \ldots, n\}?
Binomial Theorem: $x = 1$

**Theorem:** $2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$

**Proof:** How many subsets of $\{1, \ldots, n\}$?

Construct a subset with sequence of $n$ choices:
Binomial Theorem: $x = 1$

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Construct a subset with sequence of $n$ choices:
- element $i$ is in or is not in the subset: 2 poss.
Binomial Theorem: $x = 1$

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**Proof:** How many subsets of $\{1, \ldots, n\}$?
Construct a subset with sequence of $n$ choices:
- element $i$ is in or is not in the subset: 2 poss.
First rule of counting: $2 \times 2 \cdots \times 2 = 2^n$ subsets.
Binomial Theorem: $x = 1$

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How many subsets of $\{1, \ldots, n\}$?
- $\binom{n}{i}$ ways to choose $i$ elts of $\{1, \ldots, n\}$. 
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Disjoint?
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  Disjoint? Different size if in different group.
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Sum over \( i \) to get total number of subsets..
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\[ \square \]
Simple Inclusion/Exclusion

**Sum Rule:** For disjoint sets $S$ and $T$, $|S \cup T| = |S| + |T|$
Used to reason about all subsets
   by adding number of subsets of size 1, 2, 3, . . .
Simple Inclusion/Exclusion

**Sum Rule:** For disjoint sets $S$ and $T$, \(|S \cup T| = |S| + |T|\)
Used to reason about all subsets
   by adding number of subsets of size 1, 2, 3,…

Also reasoned about subsets that contained
   or didn’t contain an element. (E.g., first element, first $i$ elements.)
Simple Inclusion/Exclusion

**Sum Rule:** For disjoint sets $S$ and $T$, $|S \cup T| = |S| + |T|$
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**Inclusion/Exclusion Rule:**
For any $S$ and $T$, $|S \cup T| = |S| + |T| - |S \cap T|$.
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In $T. \implies |T|$
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---

![Venn Diagram](image-url)
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**Inclusion/Exclusion Rule:**
For any $S$ and $T$, $|S \cup T| = |S| + |T| - |S \cap T|$.

Elements in $S \cap T$ are counted twice.
**Simple Inclusion/Exclusion**

**Sum Rule:** For disjoint sets $S$ and $T$, $|S \cup T| = |S| + |T|

Used to reason about all subsets by adding number of subsets of size 1, 2, 3, ...

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**Inclusion/Exclusion Rule:**
For any $S$ and $T$, $|S \cup T| = |S| + |T| - |S \cap T|$. 

In $T$. $\implies |T|$

In $S$. $\implies + |S|$

Elements in $S \cap T$ are counted twice. Subtract. $\implies - |S \cap T|$
Simple Inclusion/Exclusion

**Sum Rule:** For disjoint sets $S$ and $T$, $|S \cup T| = |S| + |T|$

Used to reason about all subsets by adding number of subsets of size 1, 2, 3, ... 

Also reasoned about subsets that contained or didn’t contain an element. (E.g., first element, first $i$ elements.)

**Inclusion/Exclusion Rule:**
For any $S$ and $T$, $|S \cup T| = |S| + |T| - |S \cap T|$.

Elements in $S \cap T$ are counted twice. Subtract $|S \cap T|$.

$|S \cup T| = |S| + |T| - |S \cap T|$
Using Inclusion/Exclusion.

**Inclusion/Exclusion Rule:** For any $S$ and $T$, 
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**Inclusion/Exclusion Rule:** For any $S$ and $T$, 
\[ |S \cup T| = |S| + |T| - |S \cap T|. \]

**Example:** How many 10-digit phone numbers have 7 as their first or second digit?
Using Inclusion/Exclusion.

**Inclusion/Exclusion Rule:** For any $S$ and $T$,

$$|S \cup T| = |S| + |T| - |S \cap T|.$$  

**Example:** How many 10-digit phone numbers have 7 as their first or second digit?

$S =$ phone numbers with 7 as first digit.

$T =$ phone numbers with 7 as second digit.

$S \cap T =$ phone numbers with 7 as first and second digit.

Answer:

$$|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8.$$
Using Inclusion/Exclusion.

Inclusion/Exclusion Rule: For any \( S \) and \( T \),
\[
|S \cup T| = |S| + |T| - |S \cap T|.
\]

Example: How many 10-digit phone numbers have 7 as their first or second digit?
\( S = \) phone numbers with 7 as first digit. \( |S| = 10^9 \)
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$S =$ phone numbers with 7 as first digit. $|S| = 10^9$

$T =$ phone numbers with 7 as second digit.
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$S =$ phone numbers with 7 as first digit. $|S| = 10^9$

$T =$ phone numbers with 7 as second digit. $|T| = 10^9$.

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Answer: $|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$. 
Summary.

First Rule of counting:

Objects from a sequence of choices:

\[ n_1 \times n_2 \times \cdots \times n_k \text{ objects.} \]

Second Rule of counting:

If order does not matter.

Count with order.

Divide by number of orderings/sorted object.

Typically:

\[ \binom{n+k-1}{k}. \]

Stars and Bars:

Sample \( k \) objects with replacement from \( n \).

Order doesn't matter.

Typically:

\[ \binom{n+k-1}{k}. \]

Inclusion/Exclusion: two sets of objects.

Add number of each subtract intersection of sets.

Sum Rule: If disjoint just add.

Combinatorial Proofs: Identity from counting same in two ways.

Pascal's Triangle Example:

\[ \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}. \]

RHS: Number of subsets of \( n+1 \) items size \( k \).

LHS:

\[ \binom{n}{k-1} \text{ counts subsets of } n+1 \text{ items with first item.} \]

\[ \binom{n}{k} \text{ counts subsets of } n+1 \text{ items without first item.} \]

Disjoint – so add!
Summary.

First Rule of counting: Objects from a sequence of choices:
Summary.

First Rule of counting: Objects from a sequence of choices: 

\[ n_i \] possibilities for \( i \)th choice.
Summary.

First Rule of counting: Objects from a sequence of choices:

- $n_i$ possibilities for $i$th choice.
- $n_1 \times n_2 \times \cdots \times n_k$ objects.
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First Rule of counting: Objects from a sequence of choices:
- $n_i$ possibilities for $i$th choice.
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Second Rule of counting:
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First Rule of counting: Objects from a sequence of choices:
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First Rule of counting: Objects from a sequence of choices:
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Second Rule of counting: If order does not matter.
Count with order. Divide by number of orderings/sorted object.
Typically: \(^n_k\).

Stars and Bars: Sample \(k\) objects with replacement from \(n\).
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   Add number of each subtract intersection of sets.
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CS70: On to probability.

Modeling Uncertainty: Probability Space
CS70: On to probability.

Modeling Uncertainty: Probability Space

1. Key Points
2. Random Experiments
3. Probability Space
Key Points

▶ Uncertainty does not mean "nothing is known"
▶ How to best make decisions under uncertainty?
▶ Buy stocks
▶ Detect signals (transmitted bits, speech, images, radar, diseases, etc.)
▶ Control systems (Internet, airplane, robots, self-driving cars, schedule surgeries in a hospital, etc.)
▶ How to best use 'artificial' uncertainty?
▶ Play games of chance
▶ Design randomized algorithms.
▶ Probability
▶ Models knowledge about uncertainty
▶ Discovers best way to use that knowledge in making decisions
Key Points

▶ Uncertainty does not mean “nothing is known”
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The Magic of Probability

Uncertainty: vague, fuzzy, confusing, scary, hard to think about.

Probability: a precise, unambiguous, simple (!) way to think about uncertainty.

Our mission: help you discover the serenity of Probability, i.e., enable you to think clearly about uncertainty.

Your cost: focused attention and practice on examples and problems.
The Magic of Probability

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Uncertainty = Fear
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Uncertainty: vague, fuzzy, confusing, scary, hard to think about.

Probability: A precise, unambiguous, simple(!) way to think about uncertainty.

Uncertainty = Fear

Probability = Serenity
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Random Experiment: Flip one Fair Coin

- Flip a fair coin: (One flips or tosses a coin)
- Possible outcomes: Heads (H) and Tails (T)
- Likelihoods: H: 50% and T: 50%
Random Experiment: Flip one Fair Coin

Flip a fair coin:
Flip a *fair* coin: (*One flips or tosses a coin*)
Random Experiment: Flip one Fair Coin

Flip a fair coin: *(One flips or tosses a coin)*
Random Experiment: Flip one Fair Coin

Flip a fair coin: (*One flips or tosses a coin*)

- Possible outcomes:
  - Heads (H): 50% 
  - Tails (T): 50%
Random Experiment: Flip one Fair Coin

Flip a fair coin: (*One flips or tosses a coin*)

- **Possible outcomes:** Heads (*H*)
Random Experiment: Flip one Fair Coin

Flip a fair coin: *(One flips or tosses a coin)*

- Possible outcomes: Heads *(H)* and Tails *(T)*
Random Experiment: Flip one Fair Coin

Flip a **fair** coin: *(One flips or tosses a coin)*

- Possible outcomes: Heads *(H)* and Tails *(T)*
  *(One flip yields either ‘heads’ or ‘tails’.*
Random Experiment: Flip one Fair Coin

Flip a fair coin: (One flips or tosses a coin)

Possible outcomes: Heads (H) and Tails (T) (One flip yields either ‘heads’ or ‘tails’.)

Likelihoods:
Random Experiment: Flip one Fair Coin

Flip a **fair** coin: *(One flips or tosses a coin)*

- Possible outcomes: Heads *(H)* and Tails *(T)* *(One flip yields either ‘heads’ or ‘tails’).*
- Likelihoods: *(H) : 50\% and (T) : 50\%*
Random Experiment: Flip one Fair Coin

Flip a fair coin:

What do we mean by the likelihood of tails is 50%?

Two interpretations:

▶ Single coin flip: 50% chance of 'tails' [subjectivist]

▶ Many coin flips: About half yield 'tails' [frequentist]

Makes sense for many flips

▶ Question: Why does the fraction of tails converge to the same value every time?

Statistical Regularity! Deep!
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Flip a fair coin:
Random Experiment: Flip one Fair Coin

Flip a *fair* coin: model
Random Experiment: Flip one Fair Coin

Flip a **fair** coin: model

- Physical Experiment
- Probability Model

\[ \Omega = \{ H, T \} \]

\[ \Pr[H] = 0.5, \quad \Pr[T] = 0.5. \]
Random Experiment: Flip one Fair Coin

Flip a *fair* coin: model

- The physical experiment is complex.

![Physical Experiment](image1.png)

![Probability Model](image2.png)
Random Experiment: Flip one Fair Coin

Flip a **fair** coin: model

- The physical experiment is complex. (Shape, density, initial momentum and position, ...)

![Physical Experiment](image1)

![Probability Model](image2)

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Flip a fair coin: model

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  - A set $\Omega$ of outcomes: $\Omega = \{H, T\}$. 
Random Experiment: Flip one Fair Coin

Flip a *fair* coin: model

The physical experiment is complex. (Shape, density, initial momentum and position, ...)

The Probability model is simple:

- A set $\Omega$ of outcomes: $\Omega = \{H, T\}$.
- A probability assigned to each outcome: $Pr[H] = 0.5, Pr[T] = 0.5$. 
Random Experiment: Flip one Unfair Coin

Flip an unfair (biased, loaded) coin:

▶ Possible outcomes: Heads (H) and Tails (T)
▶ Likelihoods:
  - H: \( p \in (0, 1) \) and T: \( 1 - p \)

▶ Frequentist Interpretation: Flip many times ⇒ Fraction \( 1 - p \) of tails

▶ Question: How can one figure out \( p \)?

Flip many times

▶ Tautology? No: Statistical regularity!
Random Experiment: Flip one Unfair Coin

Flip an unfair (biased, loaded) coin:
Random Experiment: Flip one Unfair Coin

Flip an unfair (biased, loaded) coin:

- H: 45%
- T: 55%
Random Experiment: Flip one Unfair Coin

Flip an unfair (biased, loaded) coin:

Possible outcomes:

- H: 45%
- T: 55%
Random Experiment: Flip one Unfair Coin

Flip an unfair (biased, loaded) coin:

- Possible outcomes: Heads ($H$) and Tails ($T$)

- Likelihoods:
  - $H$: $p \in (0, 1)$
  - $T$: $1 - p$

- Frequentist Interpretation:
  Flip many times $\Rightarrow$ Fraction $1 - p$ of tails

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- Likelihoods: $H : p \in (0, 1)$ and $T : 1 - p$

H: 45%
T: 55%
Random Experiment: Flip one Unfair Coin

Flip an **unfair** (biased, loaded) coin:

- **Possible outcomes**: Heads (\(H\)) and Tails (\(T\))
- **Likelihoods**: \(H : p \in (0, 1)\) and \(T : 1 - p\)
- **Frequentist Interpretation**: 

\[
\text{H: 45\%} \\
\text{T: 55\%}
\]
Random Experiment: Flip one Unfair Coin

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Flip an unfair (biased, loaded) coin: model
Random Experiment: Flip one Unfair Coin

Flip an **unfair** (biased, loaded) coin: model

![Physical Experiment](image)

![Probability Model](image)
Flip Two Fair Coins

Possible outcomes: \{HH, HT, TH, TT\} \equiv \{H, T\}^2.

Note: \(A \times B := \{(a, b) | a \in A, b \in B\}\) and \(A^2 := A \times A\).

Likelihoods: \(\frac{1}{4}\) each.
Flip Two Fair Coins

Possible outcomes:

\{HH, HT, TH, TT\} \equiv \{H, T\}^2.

Note: $A \times B := \{(a, b) \mid a \in A, b \in B\}$ and $A^2 := A \times A$. Likelihoods: $1/4$ each.
Flip Two Fair Coins

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Flip Two Fair Coins

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- Likelihoods: 1/4 each.
Flip Glued Coins

Flips two coins glued together side by side:

▶ Possible outcomes: \{HH, TT\}.

▶ Likelihoods: HH: 0.5, TT: 0.5.

Note: Coins are glued so that they show the same face.
Flip Glued Coins

Flips two coins glued together side by side:

Possible outcomes: {HH, TT}.

Likelihoods: HH: 0.5, TT: 0.5.

Note: Coins are glued so that they show the same face.
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Flips two coins glued together side by side:

Possible outcomes: {HH, TT}.

Likelihoods: HH: 0.5, TT: 0.5.

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Flips two coins glued together side by side:

Possible outcomes:
Flip Glued Coins

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Flip Glued Coins

Flips two coins glued together side by side:

- Possible outcomes: \( \{HH, TT\} \).
- Likelihoods:
Flip Glued Coins

Flips two coins glued together side by side:

Possible outcomes: \{HH, TT\}.
Likelihoods: \(HH : 0.5\), \(TT : 0.5\).
Flip Glued Coins

Flips two coins glued together side by side:

- Possible outcomes: \{HH, TT\}.
- Likelihoods: \(HH : 0.5, TT : 0.5\).
- Note: Coins are glued so that they show the same face.
Flip Glued Coins

Flip two coins glued together side by side:

- Possible outcomes: \{HT, TH\}.
- Likelihoods: HT: 0.5, TH: 0.5.

Note: Coins are glued so that they show different faces.
Flip Glued Coins

Flips two coins glued together side by side:

Possible outcomes: {HT, TH}

Likelihoods: HT: 0.5, TH: 0.5.

Note: Coins are glued so that they show different faces.
Flip Glued Coins

Flips two coins glued together side by side:

- Possible outcomes: HT, TH
- Likelihoods: HT: 0.5, TH: 0.5

Note: Coins are glued so that they show different faces.
Flip Glued Coins

Flips two coins glued together side by side:

Possible outcomes:

- HT: 0.5
- TH: 0.5

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- Possible outcomes: \(\{HT, TH\}\).
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- Possible outcomes: \{HT, TH\}.
- Likelihoods:
Flip Glued Coins

Flips two coins glued together side by side:

- Possible outcomes: \{HT, TH\}.
- Likelihoods: \(HT : 0.5\), \(TH : 0.5\).
Flip Glued Coins

Flips two coins glued together side by side:

- Possible outcomes: \( \{ HT, TH \} \).
- Likelihoods: \( HT : 0.5, TH : 0.5 \).
- Note: Coins are glued so that they show different faces.
Flip two Attached Coins

Flips two coins attached by a spring:

▶ Possible outcomes: {HH, HT, TH, TT}.

▶ Likelihoods:
  - HH: 0.4
  - HT: 0.1
  - TH: 0.1
  - TT: 0.4

▶ Note: Coins are attached so that they tend to show the same face, unless the spring twists enough.
Flip two Attached Coins

Flips two coins attached by a spring:

Possible outcomes:
- HH
- HT
- TH
- TT

Likelihoods:
- HH: 0.4
- HT: 0.1
- TH: 0.1
- TT: 0.4

Note: Coins are attached so that they tend to show the same face, unless the spring twists enough.
Flip two Attached Coins

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Flip two Attached Coins

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  - HT: 0.1
  - TH: 0.1
  - TT: 0.4

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Flips two coins attached by a spring:

- **Possible outcomes:** \{HH, HT, TH, TT\}. 
Flip two Attached Coins

Flips two coins attached by a spring:

- Possible outcomes: \{HH, HT, TH, TT\}.
- Likelihoods:
Flip two Attached Coins

Flips two coins attached by a spring:

Possible outcomes: \(\{HH, HT, TH, TT\}\).

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Flipping Two Coins

Here is a way to summarize the four random experiments:

- $\Omega$ is the set of possible outcomes;
- Each outcome has a probability (likelihood);
- The probabilities are $\geq 0$ and add up to 1;
- Fair coins: $[1]$;
- Glued coins: $[3, 4]$;


Flipping Two Coins

Here is a way to summarize the four random experiments:
Flipping Two Coins

Here is a way to summarize the four random experiments:

1. **Ω**
   - TH: 0.25
   - HH: 0.25
   - TT: 0.25
   - HT: 0.25

2. **Ω**
   - TH: 0.1
   - HH: 0.4
   - TT: 0.4
   - HT: 0.1

3. **Ω**
   - TH: 0
   - HH: 0.5
   - TT: 0.5
   - HT: 0

4. **Ω**
   - TH: 0.5
   - HH: 0
   - TT: 0
   - HT: 0.5
Flipping Two Coins

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- $\Omega$ is the set of possible outcomes;

![Diagram showing four experiments with different probabilities for TT, TH, HT, and HH outcomes.]

[1] $\Omega$: TH 0.25 0.25, TT 0.25 0.25

[2] $\Omega$: TH 0.1 0.4, TT 0.4 0.1

[3] $\Omega$: TH 0 0.5, TT 0.5 0

[4] $\Omega$: TH 0.5 0, TT 0 0.5
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![Diagrams showing outcomes and probabilities for each experiment.](Diagram)
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    Spring-attached coins: [2];
Flipping Two Coins
Here is a way to summarize the four random experiments:

[1] \[\Omega = \{TH, TT, HT, HH\} \quad \text{with probabilities} \quad \{0.25, 0.25, 0.25, 0.25\} \]

[2] \[\Omega = \{TH, TT, HT, HH\} \quad \text{with probabilities} \quad \{0.1, 0.4, 0.1, 0.4\} \]

[3] \[\Omega = \{TH, TT, HT, HH\} \quad \text{with probabilities} \quad \{0, 0.5, 0.5, 0\} \]

[4] \[\Omega = \{TH, TT, HT, HH\} \quad \text{with probabilities} \quad \{0.5, 0, 0, 0.5\} \]
Flipping Two Coins

Here is a way to summarize the four random experiments:

Important remarks:
Flipping Two Coins

Here is a way to summarize the four random experiments:

![Diagram of four circles representing outcomes of flipping two coins]

Important remarks:

- Each outcome describes the two coins.
**Flipping Two Coins**

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- E.g., \( HT \) is one outcome of the experiment.
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- It is wrong to think that the outcomes are \(\{H, T\}\) and that one picks twice from that set.
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- Indeed, this viewpoint misses the relationship between the two flips.
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- Each $\omega \in \Omega$ describes one outcome of the complete experiment.
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▶ Each outcome describes the two coins.
▶ E.g., HT is one outcome of the experiment.
▶ It is wrong to think that the outcomes are \{H, T\} and that one picks twice from that set.
▶ Indeed, this viewpoint misses the relationship between the two flips.
▶ Each \(\omega \in \Omega\) describes one outcome of the complete experiment.
▶ \(\Omega\) and the probabilities specify the random experiment.
Flipping $n$ times

Flip a fair coin $n$ times (some $n \geq 1$):

Thus, $2^n$ possible outcomes.

Note: 
\[
\{TT \cdots T, TT \cdots H, \ldots, HH \cdots H\} = \{H, T\}^n.
\]

$A_n := \left\{ (a_1, \ldots, a_n) \mid a_1 \in A, \ldots, a_n \in A \right\}$.

$|A_n| = |A|^n$.

Likelihoods: $\frac{1}{2^n}$ each.
Flipping $n$ times
Flip a **fair** coin $n$ times (some $n \geq 1$):

- Possible outcomes:
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- Possible outcomes: $\{TT \cdots T, TT \cdots H, \ldots, HH \cdots H\}$.
  Thus, $2^n$ possible outcomes.
- Note: $\{TT \cdots T, TT \cdots H, \ldots, HH \cdots H\} = \{H, T\}^n$.
- $A^n := \{(a_1, \ldots, a_n) \mid a_1 \in A, \ldots, a_n \in A\}$. $|A^n| = |A|^n$.
- Likelihoods: $1/2^n$ each.
Flipping $n$ times

Flip a fair coin $n$ times (some $n \geq 1$):

- Possible outcomes: $\{TT \cdots T, TT \cdots H, \ldots, HH \cdots H\}$. Thus, $2^n$ possible outcomes.
- Note: $\{TT \cdots T, TT \cdots H, \ldots, HH \cdots H\} = \{H, T\}^n$.
  
  $A^n := \{(a_1, \ldots, a_n) | a_1 \in A, \ldots, a_n \in A\}$. $|A^n| = |A|^n$.
- Likelihoods: $1/2^n$ each.
Roll two Dice

Roll a balanced 6-sided die twice:
Roll two Dice

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- Possible outcomes:
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- Possible outcomes: \( \{1, 2, 3, 4, 5, 6\}^2 = \{(a, b) \mid 1 \leq a, b \leq 6\} \).
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Probability Space.

1. A “random experiment”:
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Probability Space.

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2. A set of possible outcomes: \( \Omega \).
Probability Space.

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   (a) $\Omega = \{H, T\}$;
Probability Space.

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   (c) $\Omega = \{A♠ A♦ A♣ A♥ K♠, A♠ A♦ A♣ A♥ Q♠, \ldots\}$
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      $|\Omega| = \binom{52}{5}$. 
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   $|\Omega| = \left(\begin{array}{c} 52 \\ 5 \end{array}\right)$.

3. Assign a probability to each outcome: $Pr : \Omega \to [0, 1]$.
   (a) $Pr[H] = p, Pr[T] = 1 - p$ for some $p \in [0, 1]$
Probability Space.

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   (c) $Pr[\underline{A\spadesuit A\diamondsuit A\clubsuit A\heartsuit \text{K}\spadesuit}] = \cdots = 1/\binom{52}{5}$
Probability Space: formalism.

Ω is the sample space.
Probability Space: formalism.

\(\Omega\) is the **sample space**.
\(\omega \in \Omega\) is a **sample point**.
Probability Space: formalism.

\( \Omega \) is the **sample space**.
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Probability Space: formalism.

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![Sample Space Diagram](image-url)
Probability Space: Formalism.

In a **uniform probability space** each outcome $\omega$ is **equally probable**: $Pr[\omega] = \frac{1}{|\Omega|}$ for all $\omega \in \Omega$. 

Examples:

▶ Flipping two fair coins, dealing a poker hand are uniform probability spaces.

▶ Flipping a biased coin is not a uniform probability space.
Probability Space: Formalism.

In a **uniform probability space** each outcome $\omega$ is **equally probable**: $Pr[\omega] = \frac{1}{|\Omega|}$ for all $\omega \in \Omega$.

![Uniform Probability Space Diagram](image_url)
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- Flipping two fair coins, dealing a poker hand are uniform probability spaces.
In a **uniform probability space** each outcome $\omega$ is **equally probable**:

$$Pr[\omega] = \frac{1}{|\Omega|}$$

for all $\omega \in \Omega$.

**Examples:**

- Flipping two fair coins, dealing a poker hand are uniform probability spaces.
- Flipping a biased coin is not a uniform probability space.
Probability Space: Formalism

Simplest physical model of a *uniform* probability space:

A bag of identical balls, except for their color (or a label).
If the bag is well shaken, every ball is equally likely to be picked.

\[ \Omega = \{ \text{white, red, yellow, grey, purple, blue, maroon, green} \} \]

\[ \Pr[\text{blue}] = \frac{1}{8} \]
Probability Space: Formalism

Simplest physical model of a uniform probability space:

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Physical experiment

Probability model
A bag of identical balls, except for their color (or a label).
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Simplest physical model of a non-uniform probability space:
Probability Space: Formalism

Simplest physical model of a **non-uniform** probability space:

\[ \Omega = \{ \text{Red}, \text{Green}, \text{Yellow}, \text{Blue} \} \]

\[ \Pr[\omega] = \begin{cases} 
\frac{3}{10} & \text{Red} \\
\frac{4}{10} & \text{Green} \\
\frac{2}{10} & \text{Yellow} \\
\frac{1}{10} & \text{Blue} 
\end{cases} \]
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Note: Probabilities are restricted to rational numbers: \( \frac{N_k}{N} \).
Probability Space: Formalism

Physical model of a general non-uniform probability space:

- Physical experiment
- Probability model
  - Purple = 2
  - Green = 1
  - Yellow

\( \Omega = \{1, 2, 3, \ldots, N\} \)

\( \Pr[\omega] = p_\omega \)

The roulette wheel stops in sector \( \omega \) with probability \( p_\omega \).
Probability Space: Formalism

Physical model of a general **non-uniform** probability space:

- Physical experiment
- Probability model

The roulette wheel stops in sector $\omega$ with probability $p_\omega$.

$\Omega = \{1, 2, 3, \ldots, N\}$, $Pr[\omega] = p_\omega$. 

- Green = 1
- Purple = 2
- Yellow

Fraction $p_1$ of circumference

$P_r[\omega]$
Probability Space: Formalism

Physical model of a general non-uniform probability space:

The roulette wheel stops in sector $\omega$ with probability $p_\omega$. 

Physical experiment

Probability model
Probability Space: Formalism

Physical model of a general non-uniform probability space:

\[ \Omega = \{1, 2, 3, \ldots, N\}, \]

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Probability Space: Formalism

Physical model of a general non-uniform probability space:

The roulette wheel stops in sector $\omega$ with probability $p_\omega$.

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An important remark

- The random experiment selects one and only one outcome in $\Omega$. 
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- In this case, it's wrong to think that $\Omega = \{H, T\}$ and that the experiment selects two outcomes.
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- Why?
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- Why? Because this would not describe how the two coin flips are related to each other.
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- For instance, when we flip a fair coin **twice**
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- In this case, it's wrong to think that $\Omega = \{H, T\}$ and that the experiment selects two outcomes.
- Why? Because this would not describe how the two coin flips are related to each other.
- For instance, say we glue the coins side-by-side so that they face up the same way. Then one gets $HH$ or $TT$ with probability 50% each. This is not captured by ‘picking two outcomes.’
1. Random Experiment

2. Probability Space: 
\[ \Omega; \Pr[\omega] \in [0, 1]; \sum_{\omega} \Pr[\omega] = 1. \]

3. Uniform Probability Space: 
\[ \Pr[\omega] = \frac{1}{|\Omega|} \text{ for all } \omega \in \Omega. \]
Lecture 15: Summary

Modeling Uncertainty: Probability Space

1. Random Experiment
Lecture 15: Summary

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