Total Probability: Intuition, pictures, inference.
Total Probability: Intuition, pictures, inference.
Bayes Rule.
Today

Total Probability: Intuition, pictures, inference.
Bayes Rule.
Balls in Bins.
Today

Total Probability: Intuition, pictures, inference.
Bayes Rule.
Balls in Bins.
Birthday Paradox
Today

Total Probability: Intuition, pictures, inference.
Bayes Rule.
Balls in Bins.
Birthday Paradox
Coupon Collector
Independence

**Definition:** Two events $A$ and $B$ are independent if

$$\Pr[A \cap B] = \Pr[A] \Pr[B].$$
Independence

Definition: Two events $A$ and $B$ are independent if

$$Pr[A \cap B] = Pr[A]Pr[B].$$
Independence

**Definition:** Two events $A$ and $B$ are **independent** if

$$Pr[A \cap B] = Pr[A]Pr[B].$$

**Examples:**

- When rolling two dice, $A = \text{sum is } 7$ and $B = \text{red die is } 1$ are independent; $Pr[A \cap B] = \frac{1}{36}$, $Pr[A]Pr[B] = \left(\frac{1}{6}\right) \left(\frac{1}{6}\right)$.

- When rolling two dice, $A = \text{sum is } 3$ and $B = \text{red die is } 1$ are not independent; $Pr[A \cap B] = \frac{1}{36}$, $Pr[A]Pr[B] = \left(\frac{2}{36}\right) \left(\frac{1}{6}\right)$.

- When flipping coins, $A = \text{coin 1 yields heads}$ and $B = \text{coin 2 yields tails}$ are independent; $Pr[A \cap B] = \frac{1}{4}$, $Pr[A]Pr[B] = \left(\frac{1}{2}\right) \left(\frac{1}{2}\right)$.

- When throwing 3 balls into 3 bins, $A = \text{bin 1 is empty}$ and $B = \text{bin 2 is empty}$ are not independent; $Pr[A \cap B] = \frac{1}{27}$, $Pr[A]Pr[B] = \left(\frac{8}{27}\right) \left(\frac{8}{27}\right)$.
Independence

Definition: Two events $A$ and $B$ are independent if

$$Pr[A \cap B] = Pr[A]Pr[B].$$

Examples:

- When rolling two dice, $A =$ sum is 7 and $B =$ red die is 1 are independent; $Pr[A \cap B] = \frac{1}{36}$, $Pr[A]Pr[B] = \left(\frac{1}{6}\right) \left(\frac{1}{6}\right)$.

- When rolling two dice, $A =$ sum is 3 and $B =$ red die is 1 are not independent; $Pr[A \cap B] = \frac{1}{36}$, $Pr[A]Pr[B] = \left(\frac{2}{36}\right) \left(\frac{1}{6}\right)$.

- When flipping coins, $A =$ coin 1 yields heads and $B =$ coin 2 yields tails are independent; $Pr[A \cap B] = \frac{1}{4}$, $Pr[A]Pr[B] = \left(\frac{1}{2}\right) \left(\frac{1}{2}\right)$.

- When throwing 3 balls into 3 bins, $A =$ bin 1 is empty and $B =$ bin 2 is empty are not independent; $Pr[A \cap B] = \frac{1}{27}$, $Pr[A]Pr[B] = \left(\frac{8}{27}\right) \left(\frac{8}{27}\right)$. 
Definition: Two events $A$ and $B$ are independent if $$Pr[A \cap B] = Pr[A]Pr[B].$$

Examples:

- When rolling two dice, $A = \text{sum is 7}$ and $B = \text{red die is 1}$ are independent;
- When rolling two dice, $A = \text{sum is 3}$ and $B = \text{red die is 1}$ are not independent;
- When flipping coins, $A = \text{coin 1 yields heads}$ and $B = \text{coin 2 yields tails}$ are independent;
- When throwing 3 balls into 3 bins, $A = \text{bin 1 is empty}$ and $B = \text{bin 2 is empty}$ are not independent;
Independence

**Definition:** Two events $A$ and $B$ are independent if

$$Pr[A \cap B] = Pr[A]Pr[B].$$

Examples:

- When rolling two dice, $A = \text{sum is 7}$ and $B = \text{red die is 1}$ are independent; $Pr[A \cap B] = \frac{1}{36}$, $Pr[A]Pr[B] = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right)$.
- When rolling two dice, $A = \text{sum is 3}$ and $B = \text{red die is 1}$ are not independent; $Pr[A \cap B] = \frac{1}{36}$, $Pr[A]Pr[B] = \left(\frac{2}{36}\right)\left(\frac{1}{6}\right)$.
- When flipping coins, $A = \text{coin 1 yields heads}$ and $B = \text{coin 2 yields tails}$ are independent; $Pr[A \cap B] = \frac{1}{4}$, $Pr[A]Pr[B] = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$.
- When throwing 3 balls into 3 bins, $A = \text{bin 1 is empty}$ and $B = \text{bin 2 is empty}$ are not independent; $Pr[A \cap B] = \frac{1}{27}$, $Pr[A]Pr[B] = \left(\frac{8}{27}\right)\left(\frac{8}{27}\right)$.
Independence

**Definition:** Two events $A$ and $B$ are independent if

$$Pr[A \cap B] = Pr[A]Pr[B].$$

Examples:

- When rolling two dice, $A =$ sum is 7 and $B =$ red die is 1 are independent; $Pr[A \cap B] = \frac{1}{36}, Pr[A]Pr[B] = \left(\frac{1}{6}\right) \left(\frac{1}{6}\right)$.

- When rolling two dice, $A =$ sum is 3 and $B =$ red die is 1 are
Independence

Definition: Two events $A$ and $B$ are independent if

$$Pr[A \cap B] = Pr[A]Pr[B].$$

Examples:

- When rolling two dice, $A =$ sum is 7 and $B =$ red die is 1 are independent; $Pr[A \cap B] = \frac{1}{36}$, $Pr[A]Pr[B] = \left( \frac{1}{6} \right) \left( \frac{1}{6} \right)$.
- When rolling two dice, $A =$ sum is 3 and $B =$ red die is 1 are not independent;
**Independence**

**Definition:** Two events $A$ and $B$ are **independent** if

$$Pr[A \cap B] = Pr[A]Pr[B].$$

Examples:

- When rolling two dice, $A =$ sum is 7 and $B =$ red die is 1 are independent; $Pr[A \cap B] = \frac{1}{36}, \ Pr[A]Pr[B] = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right)$.

- When rolling two dice, $A =$ sum is 3 and $B =$ red die is 1 are not independent; $Pr[A \cap B] = \frac{1}{36}, \ Pr[A]Pr[B] = \left(\frac{2}{36}\right)\left(\frac{1}{6}\right)$.
Independence

**Definition:** Two events $A$ and $B$ are **independent** if

$$Pr[A \cap B] = Pr[A]Pr[B].$$

Examples:

- When rolling two dice, $A = \text{sum is 7}$ and $B = \text{red die is 1}$ are independent; $Pr[A \cap B] = \frac{1}{36}$, $Pr[A]Pr[B] = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right)$.

- When rolling two dice, $A = \text{sum is 3}$ and $B = \text{red die is 1}$ are **not** independent; $Pr[A \cap B] = \frac{1}{36}$, $Pr[A]Pr[B] = \left(\frac{2}{36}\right)\left(\frac{1}{6}\right)$.

- When flipping coins, $A = \text{coin 1 yields heads}$ and $B = \text{coin 2 yields tails}$ are
Definition: Two events $A$ and $B$ are independent if

$$Pr[A \cap B] = Pr[A]Pr[B].$$

Examples:

- When rolling two dice, $A =$ sum is 7 and $B =$ red die is 1 are independent; $Pr[A \cap B] = \frac{1}{36}$, $Pr[A]Pr[B] = \left(\frac{1}{6}\right) \left(\frac{1}{6}\right)$.

- When rolling two dice, $A =$ sum is 3 and $B =$ red die is 1 are not independent; $Pr[A \cap B] = \frac{1}{36}$, $Pr[A]Pr[B] = \left(\frac{2}{36}\right) \left(\frac{1}{6}\right)$.

- When flipping coins, $A =$ coin 1 yields heads and $B =$ coin 2 yields tails are independent;
Independence

**Definition:** Two events $A$ and $B$ are **independent** if

\[ Pr[A \cap B] = Pr[A]Pr[B]. \]

Examples:

- When rolling two dice, $A =$ sum is 7 and $B =$ red die is 1 are independent; $Pr[A \cap B] = \frac{1}{36}, \; Pr[A]Pr[B] = \left( \frac{1}{6} \right) \left( \frac{1}{6} \right)$.

- When rolling two dice, $A =$ sum is 3 and $B =$ red die is 1 are **not** independent; $Pr[A \cap B] = \frac{1}{36}, \; Pr[A]Pr[B] = \left( \frac{2}{36} \right) \left( \frac{1}{6} \right)$.

- When flipping coins, $A =$ coin 1 yields heads and $B =$ coin 2 yields tails are independent; $Pr[A \cap B] = \frac{1}{4}, \; Pr[A]Pr[B] = \left( \frac{1}{2} \right) \left( \frac{1}{2} \right)$. 
Independence

**Definition:** Two events $A$ and $B$ are independent if

$$Pr[A \cap B] = Pr[A]Pr[B].$$

Examples:

- When rolling two dice, $A =$ sum is 7 and $B =$ red die is 1 are independent; $Pr[A \cap B] = \frac{1}{36}$, $Pr[A]Pr[B] = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right)$.

- When rolling two dice, $A =$ sum is 3 and $B =$ red die is 1 are not independent; $Pr[A \cap B] = \frac{1}{36}$, $Pr[A]Pr[B] = \left(\frac{2}{36}\right)\left(\frac{1}{6}\right)$.

- When flipping coins, $A =$ coin 1 yields heads and $B =$ coin 2 yields tails are independent; $Pr[A \cap B] = \frac{1}{4}$, $Pr[A]Pr[B] = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$.

- When throwing 3 balls into 3 bins, $A =$ bin 1 is empty and $B =$ bin 2 is empty are
Independence

**Definition:** Two events $A$ and $B$ are **independent** if

$$Pr[A \cap B] = Pr[A]Pr[B].$$

Examples:

- When rolling two dice, $A =$ sum is 7 and $B =$ red die is 1 are independent; $Pr[A \cap B] = \frac{1}{36}$, $Pr[A]Pr[B] = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right)$.
- When rolling two dice, $A =$ sum is 3 and $B =$ red die is 1 are not independent; $Pr[A \cap B] = \frac{1}{36}$, $Pr[A]Pr[B] = \left(\frac{2}{36}\right)\left(\frac{1}{6}\right)$.
- When flipping coins, $A =$ coin 1 yields heads and $B =$ coin 2 yields tails are independent; $Pr[A \cap B] = \frac{1}{4}$, $Pr[A]Pr[B] = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$.
- When throwing 3 balls into 3 bins, $A =$ bin 1 is empty and $B =$ bin 2 is empty are not independent; $Pr[A \cap B] = \frac{1}{27}$, $Pr[A]Pr[B] = \left(\frac{8}{27}\right)\left(\frac{8}{27}\right)$. 
Independence

**Definition:** Two events $A$ and $B$ are **independent** if

$$Pr[A \cap B] = Pr[A] \cdot Pr[B].$$

**Examples:**

- When rolling two dice, $A = \text{sum is 7}$ and $B = \text{red die is 1}$ are independent; $Pr[A \cap B] = \frac{1}{36}$, $Pr[A] \cdot Pr[B] = \left(\frac{1}{6}\right) \left(\frac{1}{6}\right)$.

- When rolling two dice, $A = \text{sum is 3}$ and $B = \text{red die is 1}$ are not independent; $Pr[A \cap B] = \frac{1}{36}$, $Pr[A] \cdot Pr[B] = \left(\frac{2}{36}\right) \left(\frac{1}{6}\right)$.

- When flipping coins, $A = \text{coin 1 yields heads}$ and $B = \text{coin 2 yields tails}$ are independent; $Pr[A \cap B] = \frac{1}{4}$, $Pr[A] \cdot Pr[B] = \left(\frac{1}{2}\right) \left(\frac{1}{2}\right)$.

- When throwing 3 balls into 3 bins, $A = \text{bin 1 is empty}$ and $B = \text{bin 2 is empty}$ are not independent; $Pr[A \cap B] = \frac{1}{27}$, $Pr[A] \cdot Pr[B] = \left(\frac{8}{27}\right) \left(\frac{8}{27}\right)$. 
**Fact:** Two events $A$ and $B$ are independent if and only if

$$
\Pr[A | B] = \Pr[A],
$$

so that

$$
\Pr[A | B] = \Pr[A] \iff \Pr[A \cap B] = \Pr[A] \Pr[B].
$$
Fact: Two events $A$ and $B$ are independent if and only if

$$Pr[A|B] = Pr[A].$$
**Fact:** Two events $A$ and $B$ are **independent** if and only if

$$Pr[A|B] = Pr[A].$$

Indeed:
Fact: Two events $A$ and $B$ are independent if and only if
\[ Pr[A \mid B] = Pr[A]. \]
Indeed: $Pr[A \mid B] = \frac{Pr[A \cap B]}{Pr[B]}$, so that
Fact: Two events \( A \) and \( B \) are independent if and only if

\[
Pr[A|B] = Pr[A].
\]

Indeed: \( Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} \), so that

\[
Pr[A|B] = Pr[A] \iff \frac{Pr[A \cap B]}{Pr[B]} = Pr[A]
\]
Independence and conditional probability

**Fact:** Two events $A$ and $B$ are independent if and only if

$$Pr[A|B] = Pr[A].$$

Indeed: $Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}$, so that

$$Pr[A|B] = Pr[A] \iff \frac{Pr[A \cap B]}{Pr[B]} = Pr[A] \iff Pr[A \cap B] = Pr[A]Pr[B].$$
Causality vs. Correlation

Events $A$ and $B$ are **positively correlated** if

$$Pr[A \cap B] > Pr[A]Pr[B].$$

(E.g., smoking and lung cancer.)

A and B being positively correlated does not mean that A causes B or that B causes A.

Other examples:

▶ Tesla owners are more likely to be rich. That does not mean that poor people should buy a Tesla to get rich.

▶ People who go to the opera are more likely to have a good career. That does not mean that going to the opera will improve your career.

▶ Rabbits eat more carrots and do not wear glasses. Are carrots good for eyesight?
Causality vs. Correlation

Events $A$ and $B$ are positively correlated if

$$Pr[A \cap B] > Pr[A]Pr[B].$$

(E.g., smoking and lung cancer.)
Causality vs. Correlation

Events $A$ and $B$ are **positively correlated** if

$$Pr[A \cap B] > Pr[A]Pr[B].$$

(E.g., smoking and lung cancer.)

$A$ and $B$ being positively correlated does not mean that $A$ causes $B$ or that $B$ causes $A$. 

Other examples:
- Tesla owners are more likely to be rich. That does not mean that poor people should buy a Tesla to get rich.
- People who go to the opera are more likely to have a good career. That does not mean that going to the opera will improve your career.
- Rabbits eat more carrots and do not wear glasses. Are carrots good for eyesight?
Causality vs. Correlation

Events $A$ and $B$ are **positively correlated** if

$$Pr[A \cap B] > Pr[A]Pr[B].$$

(E.g., smoking and lung cancer.)

$A$ and $B$ being positively correlated does not mean that $A$ causes $B$ or that $B$ causes $A$.

Other examples:

- Tesla owners are more likely to be rich.
Causality vs. Correlation

Events $A$ and $B$ are **positively correlated** if

$$ Pr[A \cap B] > Pr[A]Pr[B]. $$

(E.g., smoking and lung cancer.)

$A$ and $B$ being positively correlated does not mean that $A$ causes $B$ or that $B$ causes $A$.

Other examples:

- Tesla owners are more likely to be rich. That does not mean that poor people should buy a Tesla to get rich.
- People who go to the opera are more likely to have a good career. That does not mean that going to the opera will improve your career.
- Rabbits eat more carrots and do not wear glasses. Are carrots good for eyesight?
Causality vs. Correlation

Events $A$ and $B$ are **positively correlated** if

$$Pr[A \cap B] > Pr[A]Pr[B].$$

(E.g., smoking and lung cancer.)

$A$ and $B$ being positively correlated does not mean that $A$ causes $B$ or that $B$ causes $A$.

Other examples:

- Tesla owners are more likely to be rich. That does not mean that poor people should buy a Tesla to get rich.

- People who go to the opera are more likely to have a good career.
Causality vs. Correlation

Events $A$ and $B$ are **positively correlated** if

$$Pr[A \cap B] > Pr[A]Pr[B].$$

(E.g., smoking and lung cancer.)

$A$ and $B$ being positively correlated does not mean that $A$ causes $B$ or that $B$ causes $A$.

Other examples:

- Tesla owners are more likely to be rich. That does not mean that poor people should buy a Tesla to get rich.
- People who go to the opera are more likely to have a good career. That does not mean that going to the opera will improve your career.
Causality vs. Correlation

Events \( A \) and \( B \) are **positively correlated** if

\[
Pr[A \cap B] > Pr[A]Pr[B].
\]

(E.g., smoking and lung cancer.)

\( A \) and \( B \) being positively correlated does not mean that \( A \) causes \( B \) or that \( B \) causes \( A \).

Other examples:

- Tesla owners are more likely to be rich. That does not mean that poor people should buy a Tesla to get rich.
- People who go to the opera are more likely to have a good career. That does not mean that going to the opera will improve your career.
- Rabbits eat more carrots and do not wear glasses.
Causality vs. Correlation

Events $A$ and $B$ are **positively correlated** if

$$Pr[A \cap B] > Pr[A]Pr[B].$$

(E.g., smoking and lung cancer.)

$A$ and $B$ being positively correlated does not mean that $A$ causes $B$ or that $B$ causes $A$.

Other examples:

- Tesla owners are more likely to be rich. That does not mean that poor people should buy a Tesla to get rich.
- People who go to the opera are more likely to have a good career. That does not mean that going to the opera will improve your career.
- Rabbits eat more carrots and do not wear glasses. Are carrots good for eyesight?
Proving Causality

Proving causality is generally difficult.

Some difficulties:

▶ A and B may be positively correlated because they have a common cause. (E.g., being a rabbit.)

▶ If B precedes A, then B is more likely to be the cause. (E.g., smoking.) However, they could have a common cause that induces B before A. (E.g., smart, CS70, Tesla.)

More about such questions later. For fun, check "N. Taleb: Fooled by randomness."
Proving causality is generally difficult. One has to eliminate external causes of correlation and be able to test the cause/effect relationship (e.g., randomized clinical trials).
Proving Causality

Proving causality is generally difficult. One has to eliminate external causes of correlation and be able to test the cause/effect relationship (e.g., randomized clinical trials).

Some difficulties:
Proving causality is generally difficult. One has to eliminate external causes of correlation and be able to test the cause/effect relationship (e.g., randomized clinical trials).

Some difficulties:

- $A$ and $B$ may be positively correlated because they have a common cause.

More about such questions later. For fun, check “N. Taleb: Fooled by randomness.”
Proving Causality

Proving causality is generally difficult. One has to eliminate external causes of correlation and be able to test the cause/effect relationship (e.g., randomized clinical trials).

Some difficulties:

- $A$ and $B$ may be positively correlated because they have a common cause. (E.g., being a rabbit.)
Proving causality is generally difficult. One has to eliminate external causes of correlation and be able to test the cause/effect relationship (e.g., randomized clinical trials).

Some difficulties:

- $A$ and $B$ may be positively correlated because they have a common cause. (E.g., being a rabbit.)

- If $B$ precedes $A$, then $B$ is more likely to be the cause. (E.g., smoking.)

For fun, check "N. Taleb: Fooled by randomness."
Proving Causality

Proving causality is generally difficult. One has to eliminate external causes of correlation and be able to test the cause/effect relationship (e.g., randomized clinical trials).

Some difficulties:

- $A$ and $B$ may be positively correlated because they have a common cause. (E.g., being a rabbit.)

- If $B$ precedes $A$, then $B$ is more likely to be the cause. (E.g., smoking.) However, they could have a common cause that induces $B$ before $A$. 

For fun, check "N. Taleb: Fooled by randomness."
Proving causality is generally difficult. One has to eliminate external causes of correlation and be able to test the cause/effect relationship (e.g., randomized clinical trials).

Some difficulties:

- $A$ and $B$ may be positively correlated because they have a common cause. (E.g., being a rabbit.)

- If $B$ precedes $A$, then $B$ is more likely to be the cause. (E.g., smoking.) However, they could have a common cause that induces $B$ before $A$. (E.g., smart, CS70, Tesla.)
Proving causality is generally difficult. One has to eliminate external causes of correlation and be able to test the cause/effect relationship (e.g., randomized clinical trials).

Some difficulties:

- $A$ and $B$ may be positively correlated because they have a common cause. (E.g., being a rabbit.)

- If $B$ precedes $A$, then $B$ is more likely to be the cause. (E.g., smoking.) However, they could have a common cause that induces $B$ before $A$. (E.g., smart, CS70, Tesla.)

More about such questions later. For fun, check “N. Taleb: Fooled by randomness.”
Assume that $\Omega$ is the union of the disjoint sets $A_1, \ldots, A_N$. 

\[ \Pr[B] = \Pr[A_1 \cap B] + \cdots + \Pr[A_N \cap B]. \]

Indeed, $B$ is the union of the disjoint sets $A_n \cap B$ for $n = 1, \ldots, N$. 

Thus, $\Pr[B] = \Pr[A_1] \Pr[B \mid A_1] + \cdots + \Pr[A_N] \Pr[B \mid A_N]$. 


Total probability
Assume that $\Omega$ is the union of the disjoint sets $A_1, \ldots, A_N$.

Then,

$$Pr[B] = Pr[A_1 \cap B] + \cdots + Pr[A_N \cap B].$$
Total probability

Assume that $\Omega$ is the union of the disjoint sets $A_1, \ldots, A_N$.

Then,

$$Pr[B] = Pr[A_1 \cap B] + \cdots + Pr[A_N \cap B].$$

Indeed, $B$ is the union of the disjoint sets $A_n \cap B$ for $n = 1, \ldots, N$. 
Assume that $\Omega$ is the union of the disjoint sets $A_1, \ldots, A_N$.

Then,

$$Pr[B] = Pr[A_1 \cap B] + \cdots + Pr[A_N \cap B].$$

Indeed, $B$ is the union of the disjoint sets $A_n \cap B$ for $n = 1, \ldots, N$. Thus,

Total probability

Assume that $\Omega$ is the union of the disjoint sets $A_1, \ldots, A_N$.

Is your coin loaded?

Your coin is fair w.p. 1/2 or such that \( Pr[H] = 0.6 \), otherwise.
Is your coin loaded?

Your coin is fair w.p. 1/2 or such that $Pr[H] = 0.6$, otherwise.

You flip your coin and it yields heads.
Is your coin loaded?

Your coin is fair w.p. 1/2 or such that $Pr[H] = 0.6$, otherwise.

You flip your coin and it yields heads.

What is the probability that it is fair?
Is your coin loaded?

Your coin is fair w.p. $1/2$ or such that $Pr[H] = 0.6$, otherwise.

You flip your coin and it yields heads.

What is the probability that it is fair?

**Analysis:**
Is your coin loaded?

Your coin is fair w.p. 1/2 or such that $Pr[H] = 0.6$, otherwise.

You flip your coin and it yields heads.

What is the probability that it is fair?

Analysis:

$$A = \text{‘coin is fair’},$$
Is your coin loaded?
Your coin is fair w.p. 1/2 or such that $Pr[H] = 0.6$, otherwise.
You flip your coin and it yields heads.
What is the probability that it is fair?

**Analysis:**

$A = \text{‘coin is fair’}, B = \text{‘outcome is heads’}$
Is your coin loaded?

Your coin is fair w.p. $1/2$ or such that $Pr[H] = 0.6$, otherwise.

You flip your coin and it yields heads.

What is the probability that it is fair?

Analysis:

$$A = 'coin is fair', B = 'outcome is heads'$$

We want to calculate $P[A|B]$. 
Is you coin loaded?

Your coin is fair w.p. 1/2 or such that $Pr[H] = 0.6$, otherwise.

You flip your coin and it yields heads.

What is the probability that it is fair?

Analysis:

$A = \text{‘coin is fair’}, B = \text{‘outcome is heads’}$

We want to calculate $P[A|B]$.

We know $P[B|A] = \frac{1}{2}$.
Is your coin loaded?

Your coin is fair w.p. 1/2 or such that $Pr[H] = 0.6$, otherwise.

You flip your coin and it yields heads.

What is the probability that it is fair?

**Analysis:**

$A = \text{‘coin is fair’}, B = \text{‘outcome is heads’}$

We want to calculate $P[A|B]$.

We know $P[B|A] = 1/2, P[B|\bar{A}] =$
Is your coin loaded?

Your coin is fair w.p. 1/2 or such that $Pr[H] = 0.6$, otherwise.

You flip your coin and it yields heads.

What is the probability that it is fair?

**Analysis:**

\[ A = \text{‘coin is fair’}, \quad B = \text{‘outcome is heads’} \]

We want to calculate $P[A|B]$.

We know $P[B|A] = 1/2$, $P[B|\bar{A}] = 0.6$, 

Thus, $P[A|B] = \frac{P[A] \cdot P[B|A]}{P[B]} = \frac{1/2 \cdot 1/2}{1/2 + 0.6} \approx 0.45$. 
Is your coin loaded?

Your coin is fair w.p. 1/2 or such that \( Pr[H] = 0.6 \), otherwise.

You flip your coin and it yields heads.

What is the probability that it is fair?

Analysis:

\[ A = \text{‘coin is fair’}, \ B = \text{‘outcome is heads’} \]

We want to calculate \( P[A|B] \).

We know \( P[B|A] = 1/2, \ P[B|\bar{A}] = 0.6, \ Pr[A] = \)
Is your coin loaded?

Your coin is fair w.p. 1/2 or such that $Pr[H] = 0.6$, otherwise.

You flip your coin and it yields heads.

What is the probability that it is fair?

**Analysis:**

$$A = \text{‘coin is fair’, } B = \text{‘outcome is heads’}$$

We want to calculate $P[A|B]$.

We know $P[B|A] = 1/2$, $P[B|\bar{A}] = 0.6$, $Pr[A] = 1/2$
Is your coin loaded?

Your coin is fair w.p. $1/2$ or such that $Pr[H] = 0.6$, otherwise.

You flip your coin and it yields heads.

What is the probability that it is fair?

**Analysis:**

$A = \text{\'coin is fair\'}, B = \text{\'outcome is heads\'}$

We want to calculate $P[A|B]$.

We know $P[B|A] = 1/2$, $P[B|\bar{A}] = 0.6$, $Pr[A] = 1/2 = Pr[\bar{A}]$
Is your coin loaded?

Your coin is fair w.p. 1/2 or such that $Pr[H] = 0.6$, otherwise.

You flip your coin and it yields heads.

What is the probability that it is fair?

**Analysis:**

\[
A = \text{‘coin is fair’}, \quad B = \text{‘outcome is heads’}
\]

We want to calculate $P[A|B]$.

We know $P[B|A] = 1/2, P[B|\bar{A}] = 0.6, Pr[A] = 1/2 = Pr[\bar{A}]$

Now,

\[
Pr[B] = Pr[A \cap B] + Pr[\bar{A} \cap B] =
\]
Is your coin loaded?
Your coin is fair w.p. 1/2 or such that $Pr[H] = 0.6$, otherwise.
You flip your coin and it yields heads.
What is the probability that it is fair?

Analysis:

$A = \text{‘coin is fair’}, B = \text{‘outcome is heads’}$

We want to calculate $P[A|B]$.

We know $P[B|A] = 1/2$, $P[B|\overline{A}] = 0.6$, $Pr[A] = 1/2 = Pr[\overline{A}]$

Now,

$$Pr[B] = Pr[A \cap B] + Pr[\overline{A} \cap B] = Pr[A]Pr[B|A] + Pr[\overline{A}]Pr[B|\overline{A}]$$
Is your coin loaded?
Your coin is fair w.p. $1/2$ or such that $Pr[H] = 0.6$, otherwise.
You flip your coin and it yields heads.
What is the probability that it is fair?

Analysis:

$A = \text{`coin is fair'}, B = \text{`outcome is heads'}$

We want to calculate $P[A|B]$.
We know $P[B|A] = 1/2, P[B|\bar{A}] = 0.6, Pr[A] = 1/2 = Pr[\bar{A}]$
Now,

$$Pr[B] = Pr[A \cap B] + Pr[\bar{A} \cap B] = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]$$
$$= (1/2)(1/2) + (1/2)0.6 = 0.55.$$
Is your coin loaded?

Your coin is fair w.p. 1/2 or such that $Pr[H] = 0.6$, otherwise.

You flip your coin and it yields heads.

What is the probability that it is fair?

**Analysis:**

$A = \text{‘coin is fair’}, B = \text{‘outcome is heads’}$

We want to calculate $P[A|B]$.

We know $P[B|A] = 1/2, P[B|\bar{A}] = 0.6, Pr[A] = 1/2 = Pr[\bar{A}]$

Now,

$$Pr[B] = Pr[A \cap B] + Pr[\bar{A} \cap B] = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]$$

$$= (1/2)(1/2) + (1/2)0.6 = 0.55.$$  

Thus,

$$Pr[A|B] = \frac{Pr[A]Pr[B|A]}{Pr[B]} = \frac{(1/2)(1/2)}{(1/2)(1/2) + (1/2)0.6} \approx 0.45.$$
Is your coin loaded?

A picture:
Is your coin loaded?

A picture:
Imagine 100 situations, among which
\[ m := 100 \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \] 
are such that \( A \) and \( B \) occur and
\[ n := 100 \left( \frac{1}{2} \right) (0.6) \] 
are such that \( \bar{A} \) and \( B \) occur.
Is your coin loaded?

Imagine 100 situations, among which
\[ m := 100 \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \] are such that \( A \) and \( B \) occur and
\[ n := 100 \left( \frac{1}{2} \right) (0.6) \] are such that \( \bar{A} \) and \( B \) occur.

Thus, among the \( m + n \) situations where \( B \) occurred, there are \( m \) where \( A \) occurred.
Imagine 100 situations, among which $m := 100(1/2)(1/2)$ are such that $A$ and $B$ occur and $n := 100(1/2)(0.6)$ are such that $\overline{A}$ and $B$ occur.

Thus, among the $m + n$ situations where $B$ occurred, there are $m$ where $A$ occurred.

Hence,

$$Pr[A|B] = \frac{m}{m + n} = \frac{(1/2)(1/2)}{(1/2)(1/2) + (1/2)0.6}.$$
Bayes Rule

A general picture: We imagine that there are $N$ possible causes $A_1, \ldots, A_N$. 
Bayes Rule

A general picture: We imagine that there are $N$ possible causes $A_1, \ldots, A_N$. 

\[ p_n = Pr[A_n] \]

\[ q_n = Pr[B|A_n] \]

$A_1, \ldots, A_N$ disjoint

$A_1 \cup \cdots \cup A_N = \Omega$
Bayes Rule

A general picture: We imagine that there are $N$ possible causes $A_1, \ldots, A_N$.

Imagine 100 situations, among which $100p_nq_n$ are such that $A_n$ and $B$ occur, for $n = 1, \ldots, N$.

Thus, among the $100 \sum_{m} p_mq_m$ situations where $B$ occurred, there are $100p_nq_n$ where $A_n$ occurred.
Bayes Rule

A general picture: We imagine that there are $N$ possible causes $A_1, \ldots, A_N$.

Imagine 100 situations, among which $100p_nq_n$ are such that $A_n$ and $B$ occur, for $n = 1, \ldots, N$.

Thus, among the $100 \sum_m p_m q_m$ situations where $B$ occurred, there are $100p_nq_n$ where $A_n$ occurred.

Hence,

$$Pr[A_n|B] = \frac{p_nq_n}{\sum_m p_m q_m}.$$
Conditional Probability: Pictures

Left: $A$ and $B$ are independent. $\Pr[B] = b$; $\Pr[B | A] = b$.

Middle: $A$ and $B$ are positively correlated. $\Pr[B | A] = b_1 > \Pr[B | \bar{A}] = b_2$. Note: $\Pr[B] \in (b_2, b_1)$.

Right: $A$ and $B$ are negatively correlated. $\Pr[B | A] = b_1 < \Pr[B | \bar{A}] = b_2$. Note: $\Pr[B] \in (b_1, b_2)$. 
Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square
Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

- **Left:** $A$ and $B$ are independent. \( \Pr[B] = b; \Pr[B|A] = b \).

- **Middle:** $A$ and $B$ are positively correlated. \( \Pr[B|A] = b_1 > \Pr[B|\bar{A}] = b_2 \).

- **Right:** $A$ and $B$ are negatively correlated. \( \Pr[B|A] = b_1 < \Pr[B|\bar{A}] = b_2 \).

Note: \( \Pr[B] \in (b_2, b_1) \).
Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

- Left: \( A \) and \( B \) are independent.

\[ \Pr[B] = \frac{1}{2}; \quad \Pr[B | A] = \frac{1}{2} \]

\[ \Pr[B | \overline{A}] = \frac{1}{2} \]

\( \Pr[B] \in (\frac{1}{2}, 1) \)

- Middle: \( A \) and \( B \) are positively correlated.

\[ \Pr[B | A] > \Pr[B | \overline{A}] \]

\( \Pr[B] \in (\frac{1}{2}, \frac{3}{4}) \)

- Right: \( A \) and \( B \) are negatively correlated.

\[ \Pr[B | A] < \Pr[B | \overline{A}] \]

\( \Pr[B] \in (\frac{1}{4}, \frac{1}{2}) \)
Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

▶ Left: $A$ and $B$ are independent. $Pr[B] =$

▶ Middle: $A$ and $B$ are positively correlated. $Pr[B | A] =$

Note: $Pr[B] \in (b_2, b_1)$.

▶ Right: $A$ and $B$ are negatively correlated. $Pr[B | A] =$

Note: $Pr[B] \in (b_1, b_2)$. 

Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

- **Left:** $A$ and $B$ are independent. $Pr[B] = b$;

- **Middle:** $A$ and $B$ are positively correlated. $Pr[B | A] > Pr[B | \bar{A}]$.

- **Right:** $A$ and $B$ are negatively correlated. $Pr[B | A] < Pr[B | \bar{A}]$. Note: $Pr[B] \in (b_2, b_1)$. 

![Conditional Probability Diagrams](image-url)
Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

- **Left:** $A$ and $B$ are independent. $Pr[B] = b; Pr[B|A] =$

- **Middle:** $A$ and $B$ are positively correlated. $Pr[B|A] = b_1 > Pr[B|\bar{A}] = b_2$. Note: $Pr[B] \in (b_2, b_1)$.

- **Right:** $A$ and $B$ are negatively correlated. $Pr[B|A] = b_1 < Pr[B|\bar{A}] = b_2$. Note: $Pr[B] \in (b_1, b_2)$. 
Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

- **Left:** $A$ and $B$ are independent. $\Pr[B] = b; \Pr[B|A] = b$.
Illustrations: Pick a point uniformly in the unit square

- **Left:** $A$ and $B$ are independent. $Pr[B] = b; Pr[B|A] = b$.
- **Middle:** $A$ and $B$ are positively correlated. $Pr[B|A] > Pr[B|\bar{A}] = b_2$.
- **Right:** $A$ and $B$ are negatively correlated. $Pr[B|A] < Pr[B|\bar{A}] = b_2$.
Illustrations: Pick a point uniformly in the unit square

- Left: $A$ and $B$ are independent. $Pr[B] = b; Pr[B|A] = b.$
- Middle: $A$ and $B$ are positively correlated.
- Right: $A$ and $B$ are negatively correlated.
Illustrations: Pick a point uniformly in the unit square

- **Left:** $A$ and $B$ are independent. $Pr[B] = b; Pr[B|A] = b$.
- **Middle:** $A$ and $B$ are positively correlated. $Pr[B|A] =$
Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

- **Left:** $A$ and $B$ are independent. $Pr[B] = b; Pr[B|A] = b$.

- **Middle:** $A$ and $B$ are positively correlated. $Pr[B|A] = b_1 > Pr[B|\bar{A}] = \ldots$

- **Right:** $A$ and $B$ are negatively correlated. $Pr[B|A] = b_1 < Pr[B|\bar{A}] = \ldots$
Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

- **Left:** $A$ and $B$ are independent. $Pr[B] = b; Pr[B|A] = b$.
- **Middle:** $A$ and $B$ are positively correlated.
  $Pr[B|A] = b_1 > Pr[B|\bar{A}] = b_2$. 

- **Right:** $A$ and $B$ are negatively correlated.
  $Pr[B|A] = b_1 < Pr[B|\bar{A}] = b_2$. 

Note: $Pr[B] \in (b_2, b_1)$.
Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

- **Left**: $A$ and $B$ are independent. $Pr[B] = b; Pr[B|A] = b$.

- **Middle**: $A$ and $B$ are positively correlated. $Pr[B|A] = b_1 > Pr[B|\bar{A}] = b_2$. Note: $Pr[B] \in (b_2, b_1)$.

- **Right**: $A$ and $B$ are negatively correlated. $Pr[B|A] = b_1 < Pr[B|\bar{A}] = b_2$. Note: $Pr[B] \in (b_1, b_2)$. 
Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

Left: A and B are independent. $Pr[B] = b; Pr[B|A] = b$.

Middle: A and B are positively correlated.
$Pr[B|A] = b_1 > Pr[B|\bar{A}] = b_2$. Note: $Pr[B] \in (b_2, b_1)$.

Right: A and B are
Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

- **Left:** $A$ and $B$ are independent. $Pr[B] = b; Pr[B|A] = b$.
- **Middle:** $A$ and $B$ are positively correlated. $Pr[B|A] = b_1 > Pr[B|\bar{A}] = b_2$. Note: $Pr[B] \in (b_2, b_1)$.
- **Right:** $A$ and $B$ are negatively correlated.
Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

▶ Left: $A$ and $B$ are independent. $Pr[B] = b; Pr[B|A] = b$.

▶ Middle: $A$ and $B$ are positively correlated.
$Pr[B|A] = b_1 > Pr[B|\bar{A}] = b_2$. Note: $Pr[B] \in (b_2, b_1)$.

▶ Right: $A$ and $B$ are negatively correlated.
$Pr[B|A] = b_1 < Pr[B|\bar{A}] = b_2$. 
Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

- **Left:** $A$ and $B$ are independent. $Pr[B] = b; Pr[B|A] = b$.

- **Middle:** $A$ and $B$ are positively correlated.
  $Pr[B|A] = b_1 > Pr[B|\overline{A}] = b_2$. Note: $Pr[B] \in (b_2, b_1)$.

- **Right:** $A$ and $B$ are negatively correlated.
  $Pr[B|A] = b_1 < Pr[B|\overline{A}] = b_2$. Note: $Pr[B] \in (b_1, b_2)$. 
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[ \Pr[A] = 0.5; \quad \Pr[\bar{A}] = 0.5 \]

\[
\Pr[B|A] = 0.5; \quad \Pr[B|\bar{A}] = 0.6
\]

\[
\Pr[A \cap B] = 0.5 \times 0.5
\]

\[
\Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6 = \Pr[A] \Pr[B|A] + \Pr[\bar{A}] \Pr[B|\bar{A}]
\]

\[ \approx 0.46 \]

fraction of \( B \) that is inside \( A \)
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then \( \Pr[A] = 0.5; \Pr[\bar{A}] = 0.5 \)

\[ \Pr[B|A] = 0.5; \Pr[B|\bar{A}] = 0.6; \Pr[A \cap B] = 0.5 \times 0.5 = \Pr[A] \Pr[B|A] + \Pr[\bar{A}] \Pr[B|\bar{A}] \approx 0.46 = \text{fraction of } B \text{ that is inside } A \]
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]

\[ \Pr[A] = \]
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[ Pr[A] = 0.5; \]
Pick a point uniformly at random in the unit square. Then

$$Pr[A] = 0.5; Pr[\bar{A}] =$$
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

$$Pr[A] = 0.5; \ Pr[\bar{A}] = 0.5$$
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[ Pr[A] = 0.5; \quad Pr[\bar{A}] = 0.5 \]

\[ Pr[B|A] = \]
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

$$Pr[A] = 0.5; \quad Pr[\bar{A}] = 0.5$$
$$Pr[B|A] = 0.5;$$
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[ Pr[A] = 0.5; \ Pr[\bar{A}] = 0.5 \]
\[ Pr[B|A] = 0.5; \ Pr[B|\bar{A}] = \]
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[ Pr[A] = 0.5; Pr[\bar{A}] = 0.5 \]

\[ Pr[B|A] = 0.5; Pr[B|\bar{A}] = 0.6; \]
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[ Pr[A] = 0.5; \ Pr[\bar{A}] = 0.5 \]
\[ Pr[B|A] = 0.5; \ Pr[B|\bar{A}] = 0.6; \ Pr[A \cap B] = \]
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[ Pr[A] = 0.5; \quad Pr[\bar{A}] = 0.5 \]

\[ Pr[B|A] = 0.5; \quad Pr[B|\bar{A}] = 0.6; \quad Pr[A \cap B] = 0.5 \times 0.5 \]
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[ Pr[A] = 0.5; \quad Pr[\bar{A}] = 0.5 \]
\[ Pr[B|A] = 0.5; \quad Pr[B|\bar{A}] = 0.6; \quad Pr[A \cap B] = 0.5 \times 0.5 \]
\[ Pr[B] = \]

\[ \text{fraction of } B \text{ that is inside } A \]
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[ Pr[A] = 0.5; \ Pr[\bar{A}] = 0.5 \]
\[ Pr[B|A] = 0.5; \ Pr[B|\bar{A}] = 0.6; \ Pr[A \cap B] = 0.5 \times 0.5 \]
\[ Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6 \]
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

$$Pr[A] = 0.5; \ Pr[\overline{A}] = 0.5$$

$$Pr[B|A] = 0.5; \ Pr[B|\overline{A}] = 0.6; \ Pr[A \cap B] = 0.5 \times 0.5$$

$$Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6 = Pr[A]Pr[B|A] + Pr[\overline{A}]Pr[B|\overline{A}]$$
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[ Pr[A] = 0.5; \ Pr[\bar{A}] = 0.5 \]
\[ Pr[B|A] = 0.5; \ Pr[B|\bar{A}] = 0.6; \ Pr[A \cap B] = 0.5 \times 0.5 \]
\[ Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6 = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}] \]
\[ Pr[A|B] = \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.5 \times 0.6} \]
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[
Pr[A] = 0.5; \ Pr[\bar{A}] = 0.5
\]

\[
Pr[B|A] = 0.5; \ Pr[B|\bar{A}] = 0.6; \ Pr[A \cap B] = 0.5 \times 0.5
\]

\[
Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6 = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]
\]

\[
Pr[A|B] = \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.5 \times 0.6} = \frac{Pr[A]Pr[B|A]}{Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]}
\]
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[ Pr[A] = 0.5; \Pr[\bar{A}] = 0.5 \]

\[ Pr[B|A] = 0.5; \Pr[B|\bar{A}] = 0.6; \Pr[A \cap B] = 0.5 \times 0.5 \]

\[ Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6 = \Pr[A]Pr[B|A] + \Pr[\bar{A}]Pr[B|\bar{A}] \]

\[ Pr[A|B] = \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.5 \times 0.6} = \frac{Pr[A]Pr[B|A]}{Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]} \approx 0.46 \]
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[ Pr[A] = 0.5; \quad Pr[\bar{A}] = 0.5 \]
\[ Pr[B|A] = 0.5; \quad Pr[B|\bar{A}] = 0.6; \quad Pr[A \cap B] = 0.5 \times 0.5 \]
\[ Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6 = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}] \]
\[ Pr[A|B] = \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.5 \times 0.6} = \frac{Pr[A]Pr[B|A]}{Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]} \approx 0.46 = \text{fraction of } B \text{ that is inside } A \]
Bayes: General Case

Pick a point uniformly at random in the unit square. Then

\[ \Pr[A_n] = p_n, \quad n = 1, \ldots, N \]

\[ \Pr[B | A_n] = q_n, \quad n = 1, \ldots, N; \]

\[ \Pr[A_n \cap B] = p_n q_n \]

\[ \Pr[B] = p_1 q_1 + \cdots + p_N q_N \]

\[ \Pr[A_n | B] = \frac{p_n q_n}{p_1 q_1 + \cdots + p_N q_N} \]

fraction of \( B \) inside \( A_n \).
Bayes: General Case

Pick a point uniformly at random in the unit square. Then

\[ P[A_n] = p_n, \quad n = 1, \ldots, N \]

\[ P[B | A_n] = q_n, \quad n = 1, \ldots, N \]

\[ P[A_n \cap B] = p_n q_n \]

\[ P[B] = p_1 q_1 + \cdots + p_N q_N \]

\[ P[A_n | B] = \frac{p_n q_n}{p_1 q_1 + \cdots + p_N q_N} \]

Fraction of \( B \) inside \( A_n \).

Event \( B \)

\[ A_1, \ldots, A_N \] disjoint

\[ A_1 \cup \cdots \cup A_N = \Omega \]
Bayes: General Case

\begin{align*}
\text{Pr}[A_n] &= p_n, \quad n = 1, \ldots, N \\
\text{Pr}[B|A_n] &= q_n, \quad n = 1, \ldots, N \\
\text{Pr}[A_n \cap B] &= p_n \cdot q_n \\
\text{Pr}[B] &= p_1 q_1 + \cdots + p_N q_N \\
\text{Pr}[A_n|B] &= \frac{p_n q_n}{p_1 q_1 + \cdots + p_N q_N}
\end{align*}

$p_n = Pr[A_n]$  
$q_n = Pr[B|A_n]$  

$A_1, \ldots, A_N$ disjoint  
$A_1 \cup \cdots \cup A_N = \Omega$
Bayes: General Case

Pick a point uniformly at random in the unit square. Then
Bayes: General Case

Pick a point uniformly at random in the unit square. Then

\[ Pr[A_n] = p_n, \quad n = 1, \ldots, N \]
Bayes: General Case

Pick a point uniformly at random in the unit square. Then

\[ Pr[A_n] = p_n, \quad n = 1, \ldots, N \]

\[ Pr[B|A_n] = q_n, \quad n = 1, \ldots, N; \]
Bayes: General Case

Pick a point uniformly at random in the unit square. Then

\[ Pr[A_n] = p_n, \ n = 1, \ldots, N \]
\[ Pr[B|A_n] = q_n, \ n = 1, \ldots, N; \ Pr[A_n \cap B] = \]
Bayes: General Case

Pick a point uniformly at random in the unit square. Then

\[
Pr[A_n] = p_n, \ n = 1, \ldots, N
\]

\[
Pr[B|A_n] = q_n, \ n = 1, \ldots, N; \ Pr[A_n \cap B] = p_n q_n
\]
Bayes: General Case

Pick a point uniformly at random in the unit square. Then

\[ Pr[A_n] = p_n, \ n = 1, \ldots, N \]
\[ Pr[B|A_n] = q_n, \ n = 1, \ldots, N; \ Pr[A_n \cap B] = p_n q_n \]
\[ Pr[B] = p_1 q_1 + \cdots + p_N q_N \]
Bayes: General Case

Pick a point uniformly at random in the unit square. Then

\[ Pr[A_n] = p_n, \quad n = 1, \ldots, N \]
\[ Pr[B|A_n] = q_n, \quad n = 1, \ldots, N; \quad Pr[A_n \cap B] = p_n q_n \]
\[ Pr[B] = p_1 q_1 + \cdots p_N q_N \]
\[ Pr[A_n|B] = \frac{p_n q_n}{p_1 q_1 + \cdots p_N q_N} \]
Bayes: General Case

Pick a point uniformly at random in the unit square. Then

\[ Pr[A_n] = p_n, \quad n = 1, \ldots, N \]
\[ Pr[B|A_n] = q_n, \quad n = 1, \ldots, N; \quad Pr[A_n \cap B] = p_nq_n \]
\[ Pr[B] = p_1q_1 + \cdots + p_Nq_N \]
\[ Pr[A_n|B] = \frac{p_nq_n}{p_1q_1 + \cdots + p_Nq_N} = \text{fraction of } B \text{ inside } A_n. \]
Why do you have a fever?

Using Bayes' rule, we find

\[
\Pr[\text{Flu} | \text{High Fever}] = 0.15 \times 0.80 = 0.12
\]

\[
\Pr[\text{Ebola} | \text{High Fever}] = 10^{-8} \times 1 = 10^{-8}
\]

\[
\Pr[\text{Other} | \text{High Fever}] = 0.85 \times 0.10 = 0.085
\]

The values 0.12, 10^{-8}, 0.085 are the posterior probabilities.
Why do you have a fever?

Using Bayes’ rule, we find:

\[
\begin{align*}
\Pr[\text{Flu} \mid \text{High Fever}] &= 0.15 \times 0.80 \times 0.15 \\
\Pr[\text{Ebola} \mid \text{High Fever}] &= 10^{-8} \times 0.80 \\
\Pr[\text{Other} \mid \text{High Fever}] &= 0.85 \times 0.10 \\
\end{align*}
\]

The values 0.58, 5 × 10^{-8}, 0.42 are the posterior probabilities.
Why do you have a fever?

Using Bayes’ rule, we find

\[
Pr[\text{Flu}|\text{High Fever}] = \frac{0.15 \times 0.80}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.58
\]
Why do you have a fever?

Using Bayes’ rule, we find

\[
Pr[\text{Flu}|\text{High Fever}] = \frac{0.15 \times 0.80}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.58
\]

\[
Pr[\text{Ebola}|\text{High Fever}] = \frac{10^{-8} \times 1}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 5 \times 10^{-8}
\]
Why do you have a fever?

Using Bayes’ rule, we find

\[
Pr[\text{Flu}|\text{High Fever}] = \frac{0.15 \times 0.80}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.58
\]

\[
Pr[\text{Ebola}|\text{High Fever}] = \frac{10^{-8} \times 1}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 5 \times 10^{-8}
\]

\[
Pr[\text{Other}|\text{High Fever}] = \frac{0.85 \times 0.1}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.42
\]

The values 0.58, 5 \times 10^{-8}, 0.42 are the posterior probabilities.
Why do you have a fever?

Using Bayes’ rule, we find

\[ Pr[\text{Flu}|\text{High Fever}] = \frac{0.15 \times 0.80}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.58 \]

\[ Pr[\text{Ebola}|\text{High Fever}] = \frac{10^{-8} \times 1}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 5 \times 10^{-8} \]

\[ Pr[\text{Other}|\text{High Fever}] = \frac{0.85 \times 0.1}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.42 \]

The values 0.58, \(5 \times 10^{-8}\), 0.42 are the posterior probabilities.
Why do you have a fever?
Why do you have a fever?

Our “Bayes’ Square” picture:

Note that even though $\Pr[Fever | Ebola] = 1$, one has $\Pr[Ebola | Fever] \approx 0$. This example shows the importance of the prior probabilities.
Why do you have a fever?

Our “Bayes’ Square” picture:

- Flu: 58% of Fever
- Other: 42% of Fever
- Ebola: 0% of Fever

Prior probabilities:
- Flu: 0.15
- Ebola: 10^{-8}
- Other: 0.85
- High Fever: 0.10

Conditional probabilities:
- Flu: 0.80
- Ebola: 1
- Other: 0

Note that even though $\Pr[\text{Fever} | \text{Ebola}] = 1$, one has $\Pr[\text{Ebola} | \text{Fever}] \approx 0$. This example shows the importance of the prior probabilities.
Why do you have a fever?

Our “Bayes’ Square” picture:

Prior probabilities

<table>
<thead>
<tr>
<th>Condition</th>
<th>Flu</th>
<th>Ebola</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.85</td>
<td></td>
<td>0.15</td>
</tr>
<tr>
<td>10⁻⁸</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Conditional probabilities

<table>
<thead>
<tr>
<th>Condition</th>
<th>Fever</th>
<th>Flu</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.80</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note that even though $Pr[\text{Fever} | \text{Ebola}] = 1$, $Pr[\text{Ebola} | \text{Fever}] \approx 0$. This example shows the importance of the prior probabilities.
Why do you have a fever?

Our “Bayes’ Square” picture:

Note that even though $Pr[\text{Fever}|\text{Ebola}] = 1$, one has

$Pr[\text{Ebola}|\text{Fever}] \approx 0$. 

58% of Fever = Flu

$\approx 0\%$ of Fever = Ebola

42% of Fever = Other
Why do you have a fever?

Our “Bayes’ Square” picture:

\[ \Pr[\text{Fever}|\text{Ebola}] = 1, \quad \Pr[\text{Ebola}|\text{Fever}] \approx 0. \]

Note that even though \( \Pr[\text{Fever}|\text{Ebola}] = 1 \), one has \( \Pr[\text{Ebola}|\text{Fever}] \approx 0 \).

This example shows the importance of the prior probabilities.
Why do you have a fever?

We found
Why do you have a fever?

We found

\[
Pr[\text{Flu}|\text{High Fever}] \approx 0.58, \\
Pr[\text{Ebola}|\text{High Fever}] \approx 5 \times 10^{-8}, \\
Pr[\text{Other}|\text{High Fever}] \approx 0.42
\]
Why do you have a fever?

We found

\[ Pr[\text{Flu} | \text{High Fever}] \approx 0.58, \]
\[ Pr[\text{Ebola} | \text{High Fever}] \approx 5 \times 10^{-8},\]
\[ Pr[\text{Other} | \text{High Fever}] \approx 0.42\]

One says that ‘Flu’ is the Most Likely a Posteriori (MAP) cause of the high fever.
Why do you have a fever?

We found

\[
Pr[\text{Flu} | \text{High Fever}] \approx 0.58,
\]
\[
Pr[\text{Ebola} | \text{High Fever}] \approx 5 \times 10^{-8},
\]
\[
Pr[\text{Other} | \text{High Fever}] \approx 0.42
\]

One says that ‘Flu’ is the **Most Likely a Posteriori** (MAP) cause of the high fever.
‘Ebola’ is the **Maximum Likelihood Estimate** (MLE) of the cause: it causes the fever with the largest probability.
Why do you have a fever?

We found

\[ Pr[\text{Flu}|\text{High Fever}] \approx 0.58, \]
\[ Pr[\text{Ebola}|\text{High Fever}] \approx 5 \times 10^{-8}, \]
\[ Pr[\text{Other}|\text{High Fever}] \approx 0.42 \]

One says that ‘Flu’ is the **Most Likely a Posteriori** (MAP) cause of the high fever.

‘Ebola’ is the **Maximum Likelihood Estimate** (MLE) of the cause: it causes the fever with the largest probability.

Recall that

\[ p_m = Pr[A_m], \quad q_m = Pr[B|A_m], \quad Pr[A_m|B] = \frac{p_m q_m}{p_1 q_1 + \cdots + p_M q_M}. \]
Why do you have a fever?

We found

\[
Pr[\text{Flu}|\text{High Fever}] \approx 0.58, \\
Pr[\text{Ebola}|\text{High Fever}] \approx 5 \times 10^{-8}, \\
Pr[\text{Other}|\text{High Fever}] \approx 0.42
\]

One says that ‘Flu’ is the **Most Likely a Posteriori** (MAP) cause of the high fever. ‘Ebola’ is the **Maximum Likelihood Estimate** (MLE) of the cause: it causes the fever with the largest probability.

Recall that

\[
p_m = Pr[A_m], q_m = Pr[B|A_m], Pr[A_m|B] = \frac{p_m q_m}{p_1 q_1 + \cdots + p_M q_M}.
\]

Thus,

- **MAP** = value of \( m \) that maximizes \( p_m q_m \).
Why do you have a fever?

We found

\[ Pr[\text{Flu}|\text{High Fever}] \approx 0.58, \]
\[ Pr[\text{Ebola}|\text{High Fever}] \approx 5 \times 10^{-8}, \]
\[ Pr[\text{Other}|\text{High Fever}] \approx 0.42 \]

One says that ‘Flu’ is the Most Likely a Posteriori (MAP) cause of the high fever.

‘Ebola’ is the Maximum Likelihood Estimate (MLE) of the cause: it causes the fever with the largest probability.

Recall that

\[ p_m = Pr[A_m], q_m = Pr[B|A_m], Pr[A_m|B] = \frac{p_m q_m}{p_1 q_1 + \cdots + p_M q_M}. \]

Thus,

- MAP = value of \( m \) that maximizes \( p_m q_m \).
- MLE = value of \( m \) that maximizes \( q_m \).
Bayes’ Rule Operations
Bayes’ Rule Operations

[Environment]

Priors: $Pr[A_n]$

Observe $B$

Bayes’ Rule

Posterior: $Pr[A_n|B]$

Conditional: $Pr[B|A_n]$

[Model of system]
Bayes’ Rule is the canonical example of how information changes our opinions.
Thomas Bayes

Portrait used of Bayes in a 1936 book,[1] but it is doubtful whether the portrait is actually of him.[2] No earlier portrait or claimed portrait survives.

Born | c. 1701
     | London, England
Died | 7 April 1761 (aged 59)
     | Tunbridge Wells, Kent, England
Residence | Tunbridge Wells, Kent, England
Nationality | English
Known for | Bayes' theorem

A Bayesian picture of Thomas Bayes.
Testing for disease.

Random Experiment: Pick a random male.
Testing for disease.

Random Experiment: Pick a random male.
Outcomes: \((test, disease)\)
Testing for disease.

Random Experiment: Pick a random male.
Outcomes: \((test, disease)\)
\(A\) - prostate cancer.
\(B\) - positive PSA test.
Random Experiment: Pick a random male.
Outcomes: \((test, disease)\)
\(A\) - prostate cancer.
\(B\) - positive PSA test.

- \(Pr[A] = 0.0016\), (.16 % of the male population is affected.)
- \(Pr[B|A] = 0.80\) (80% chance of positive test with disease.)
- \(Pr[B|\overline{A}] = 0.10\) (10% chance of positive test without disease.)
Testing for disease.

Random Experiment: Pick a random male.
Outcomes: \((test, disease)\)
\(A\) - prostate cancer.
\(B\) - positive PSA test.

\(\cdot \quad Pr[A] = 0.0016\), (.16 % of the male population is affected.)
\(\cdot \quad Pr[B|A] = 0.80\) (80% chance of positive test with disease.)
\(\cdot \quad Pr[B|\overline{A}] = 0.10\) (10% chance of positive test without disease.)

Testing for disease.

Random Experiment: Pick a random male. 
Outcomes: \((test, disease)\) 

\(A\) - prostate cancer. 
\(B\) - positive PSA test. 

- \(Pr[A] = 0.0016\), (.16 % of the male population is affected.)
- \(Pr[B|A] = 0.80\) (80% chance of positive test with disease.)
- \(Pr[B|\bar{A}] = 0.10\) (10% chance of positive test without disease.)


Positive PSA test \((B)\). Do I have disease?
Testing for disease.

Random Experiment: Pick a random male.
Outcomes: \((test, disease)\)
\(A\) - prostate cancer.
\(B\) - positive PSA test.

\[
\begin{align*}
Pr[A] &= 0.0016, (.16 \% \text{ of the male population is affected.}) \\
Pr[B|A] &= 0.80 (80\% \text{ chance of positive test with disease.}) \\
Pr[B|\overline{A}] &= 0.10 (10\% \text{ chance of positive test without disease.})
\end{align*}
\]


Positive PSA test \((B)\). Do I have disease?

\[
Pr[A|B] ?\?
Using Bayes' rule, we find
\[ P(A|B) = 0.0016 \times 0.80 + 0.9984 \times 0.10 = 0.013 \]

A 1.3% chance of prostate cancer with a positive PSA test. Surgery anyone? Impotence... Incontinence... Death.
Using Bayes’ rule, we find

\[
P(A | B) = 0.0016 \times 0.80 + 0.9984 \times 0.10 = 0.13
\]

A 1.3% chance of prostate cancer with a positive PSA test.

Surgery anyone?

Impotence...

Incontinence...

Death.
Using Bayes’ rule, we find

$$P[A|B] = \frac{0.0016 \times 0.80}{0.0016 \times 0.80 + 0.9984 \times 0.10}$$
Bayes Rule.

Using Bayes’ rule, we find

\[
P[A|B] = \frac{0.0016 \times 0.80}{0.0016 \times 0.80 + 0.9984 \times 0.10} = 0.013.
\]
Using Bayes’ rule, we find

\[
P[A|B] = \frac{0.0016 \times 0.80}{0.0016 \times 0.80 + 0.9984 \times 0.10} = .013.
\]

A 1.3% chance of prostate cancer with a positive PSA test.
Using Bayes’ rule, we find

\[
P[A|B] = \frac{0.0016 \times 0.80}{0.0016 \times 0.80 + 0.9984 \times 0.10} = 0.013.
\]

A 1.3% chance of prostate cancer with a positive PSA test.

Surgery anyone?
Using Bayes’ rule, we find

\[
P[A|B] = \frac{0.0016 \times 0.80}{0.0016 \times 0.80 + 0.9984 \times 0.10} = 0.013.
\]

A 1.3% chance of prostate cancer with a positive PSA test.

Surgery anyone?

Impotence...
Using Bayes’ rule, we find

\[
P[A|B] = \frac{0.0016 \times 0.80}{0.0016 \times 0.80 + 0.9984 \times 0.10} = 0.013.
\]

A 1.3% chance of prostate cancer with a positive PSA test.

Surgery anyone?

Impotence...

Incontinence..
Using Bayes’ rule, we find

\[
P[A|B] = \frac{0.0016 \times 0.80}{0.0016 \times 0.80 + 0.9984 \times 0.10} = 0.013.
\]

A 1.3% chance of prostate cancer with a positive PSA test.

Surgery anyone?

Impotence...

Incontinence..

Death.
Quick Review

Events, Conditional Probability, Independence, Bayes’ Rule
Quick Review

Events, Conditional Probability, Independence, Bayes’ Rule

Key Ideas:

- **Conditional Probability:**

  \[
  Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}
  \]

- **Independence:**

  \[
  Pr[A \cap B] = Pr[A] \cdot Pr[B]
  \]

- **Bayes’ Rule:**

  \[
  Pr[A|B] = \frac{Pr[A] \cdot Pr[B|A]}{\sum_{m} Pr[A|m] \cdot Pr[B|A|m]}
  \]

  - Posterior probability
  - Prior probability
## Quick Review

<table>
<thead>
<tr>
<th>Events, Conditional Probability, Independence, Bayes’ Rule</th>
</tr>
</thead>
</table>

### Key Ideas:

- **Conditional Probability:**
  \[
  Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}
  \]

- **Independence:**
  \[
  Pr[A \cap B] = Pr[A]Pr[B].
  \]
Quick Review

Events, Conditional Probability, Independence, Bayes’ Rule

Key Ideas:

- **Conditional Probability:**
  \[
  Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}
  \]

- **Independence:** \( Pr[A \cap B] = Pr[A]Pr[B] \).

- **Bayes’ Rule:**
  \[
  Pr[A_n|B] = \frac{Pr[A_n]Pr[B|A_n]}{\sum_m Pr[A_m]Pr[B|A_m]}
  \]
Quick Review

Events, Conditional Probability, Independence, Bayes’ Rule

Key Ideas:

- **Conditional Probability:**
  \[
  Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}
  \]

- **Independence:**
  \[
  Pr[A \cap B] = Pr[A]Pr[B].
  \]

- **Bayes’ Rule:**
  \[
  Pr[A_n|B] = \frac{Pr[A_n]Pr[B|A_n]}{\sum_m Pr[A_m]Pr[B|A_m]}.
  \]

  \[
  Pr[A_n|B] = \text{posterior probability}; Pr[A_n] = \text{prior probability}.
  \]
Quick Review

Events, Conditional Probability, Independence, Bayes’ Rule

Key Ideas:

- **Conditional Probability:**
  \[
  P(A|B) = \frac{P(A \cap B)}{P(B)}
  \]

- **Independence:**
  \[
  P(A \cap B) = P(A)P(B).
  \]

- **Bayes’ Rule:**
  \[
  P(A_n|B) = \frac{P(A_n)P(B|A_n)}{\sum_m P(A_m)P(B|A_m)}.
  \]

  \[
  P(A_n|B) = \text{posterior probability}; P(A_n) = \text{prior probability}.
  \]

- All these are possible:
  \[
  P(A|B) < P(A); P(A|B) > P(A); P(A|B) = P(A).
  \]
Independence

Recall:

$A$ and $B$ are independent

$(A_2, B)$ are independent: $\Pr[A_2 | B] = 0.5 = \Pr[A_2]$. 

$(A_2, \bar{B})$ are independent: $\Pr[A_2 | \bar{B}] = 0.5 = \Pr[A_2]$. 

$(A_1, B)$ are not independent: $\Pr[A_1 | B] = 0.1 = 0.25 = \Pr[A_1]$. 

Independence

Recall:

$A$ and $B$ are independent

$\iff Pr[A \cap B] = Pr[A]Pr[B]$
Independence

Recall:

$A$ and $B$ are independent

$\iff Pr[A \cap B] = Pr[A]Pr[B]$,

$\iff Pr[A|B] = Pr[A]$. 

Consider the example below:

\begin{align*}
\begin{array}{c|c|c}
A_1 & A_2 & A_3 \\
\hline
0.1 & 0.25 & 0.15 \\
\end{array}
\begin{array}{c|c|c}
B & \overline{B} \\
\hline
0.15 & 0.25 \\
\end{array}
\end{align*}

\begin{align*}
(A_2, B) & \text{ are independent: } Pr[A_2|B] = 0.5 = Pr[A_2]. \\
(A_2, \overline{B}) & \text{ are independent: } Pr[A_2|\overline{B}] = 0.5 = Pr[A_2]. \\
(A_1, B) & \text{ are not independent: } Pr[A_1|B] = 0.1 \neq 0.25 = Pr[A_1].
\end{align*}
Independence

Recall:

\[ A \text{ and } B \text{ are independent} \iff Pr[A \cap B] = Pr[A]Pr[B] \]
\[ \iff Pr[A|B] = Pr[A]. \]

Consider the example below:

\[
\begin{array}{c|cc}
  & B & \bar{B} \\
  \hline
  A_1 & 0.1 & 0.15 \\
  A_2 & 0.25 & 0.25 \\
  A_3 & 0.15 & 0.1 \\
\end{array}
\]
**Independence**

Recall:

$A$ and $B$ are independent

$\iff Pr[A \cap B] = Pr[A]Pr[B]$

$\iff Pr[A|B] = Pr[A].$

Consider the example below:

\[(A_2, B)\] are independent:

\[(A_2, \bar{B})\] are independent:

\[(A_1, B)\] are not independent:
**Independence**

Recall:

\[ A \text{ and } B \text{ are independent } \iff \Pr[A \cap B] = \Pr[A] \Pr[B] \]
\[ \iff \Pr[A|B] = \Pr[A]. \]

Consider the example below:

\[
\begin{array}{c|c|c}
 & B & \bar{B} \\
\hline
A_1 & 0.1 & 0.15 \\
A_2 & 0.25 & 0.25 \\
A_3 & 0.15 & 0.1 \\
\end{array}
\]

\((A_2, B)\) are independent: \( \Pr[A_2|B] = 0.5 = \Pr[A_2] \).
**Independence**

Recall:

\[
A \text{ and } B \text{ are independent} \iff Pr[A \cap B] = Pr[A]Pr[B] \\
\iff Pr[A|B] = Pr[A].
\]

Consider the example below:

\[
\begin{array}{c|c|c}
& B & \bar{B} \\
\hline
A_1 & 0.1 & 0.15 \\
A_2 & 0.25 & 0.25 \\
A_3 & 0.15 & 0.1 \\
\end{array}
\]

\((A_2, B)\) are independent: \(Pr[A_2|B] = 0.5 = Pr[A_2]\).

\((A_2, \bar{B})\) are independent:
Independence

Recall:

$A$ and $B$ are independent
$\iff Pr[A \cap B] = Pr[A]Pr[B]$
$\iff Pr[A|B] = Pr[A].$

Consider the example below:

$(A_2, B)$ are independent: $Pr[A_2|B] = 0.5 = Pr[A_2].$
$(A_2, \bar{B})$ are independent: $Pr[A_2|\bar{B}] = 0.5 = Pr[A_2].$
Independence

Recall:

\( A \) and \( B \) are independent
\[ \iff Pr[A \cap B] = Pr[A]Pr[B] \]
\[ \iff Pr[A|B] = Pr[A]. \]

Consider the example below:

\((A_2, B)\) are independent: \( Pr[A_2|B] = 0.5 = Pr[A_2] \).
\((A_2, \bar{B})\) are independent: \( Pr[A_2|\bar{B}] = 0.5 = Pr[A_2] \).
\((A_1, B)\) are not independent:

\[ Pr[A_1|B] \neq Pr[A_1] \]
Independence

Recall:

$A$ and $B$ are independent

$\iff Pr[A \cap B] = Pr[A]Pr[B]$

$\iff Pr[A|B] = Pr[A]$. 

Consider the example below:

$(A_2, B)$ are independent: $Pr[A_2|B] = 0.5 = Pr[A_2]$.
$(A_2, \overline{B})$ are independent: $Pr[A_2|\overline{B}] = 0.5 = Pr[A_2]$.
$(A_1, B)$ are not independent: $Pr[A_1|B] = \frac{0.1}{0.5} = 0.2 \neq Pr[A_1] = 0.25$. 
Pairwise Independence

Flip two fair coins. Let

- $A = \text{‘first coin is H’} = \{HT, HH\}$;
- $B = \text{‘second coin is H’} = \{TH, HH\}$;
- $C = \text{‘the two coins are different’} = \{TH, HT\}$.

(A, C) are independent; (B, C) are independent; (A ∩ B, C) are not independent. ($\Pr[A ∩ B ∩ C] = 0 \neq \Pr[A ∩ B] \cdot \Pr[C]$.)

If A did not say anything about C and B did not say anything about C, then A ∩ B would not say anything about C.
Pairwise Independence

Flip two fair coins. Let

- $A = \text{‘first coin is H’} = \{HT, HH\}$;
- $B = \text{‘second coin is H’} = \{TH, HH\}$;
- $C = \text{‘the two coins are different’} = \{TH, HT\}$.

If $A$ did not say anything about $C$ and $B$ did not say anything about $C$, then $A \cap B$ would not say anything about $C$. 

$\text{Pr}[A \cap B \cap C] \neq \text{Pr}[A \cap B] \cdot \text{Pr}[C]$. 

\begin{center}
\begin{tikzpicture}
\draw[thick,blue] (0,0) rectangle (1,1);
\draw[thick,red] (1,0) rectangle (2,1);
\draw[thick,green] (0,0) -- (2,2);
\draw[thick,green] (0,1) -- (2,0);
\node at (0.5,0.5) {$C$};
\node at (1.5,0.5) {$A$};
\node at (0.5,1.5) {$B$};
\node at (0,0) {$TT$};
\node at (0,1) {$TH$};
\node at (1,0) {$C$};
\node at (1,1) {$B$};
\node at (2,0) {$HH$};
\node at (2,1) {$HT$};
\end{tikzpicture}
\end{center}
Pairwise Independence

Flip two fair coins. Let

- \( A = \text{first coin is H' = \{HT, HH\}}; \)
- \( B = \text{second coin is H' = \{TH, HH\}}; \)
- \( C = \text{the two coins are different' = \{TH, HT\}}. \)

\( A, C \) are independent;
Flip two fair coins. Let

- $A = \text{‘first coin is H’} = \{HT, HH\}$;
- $B = \text{‘second coin is H’} = \{TH, HH\}$;
- $C = \text{‘the two coins are different’} = \{TH, HT\}$.

$A, C$ are independent; $B, C$ are independent;
Pairwise Independence

Flip two fair coins. Let

- \( A = \text{first coin is H'} = \{HT, HH\}; \)
- \( B = \text{second coin is H'} = \{TH, HH\}; \)
- \( C = \text{the two coins are different'} = \{TH, HT\}. \)

\( A, C \) are independent; \( B, C \) are independent; \( A \cap B, C \) are not independent.
Pairwise Independence

Flip two fair coins. Let

- $A = \text{‘first coin is H’} = \{HT, HH\}$;
- $B = \text{‘second coin is H’} = \{TH, HH\}$;
- $C = \text{‘the two coins are different’} = \{TH, HT\}$.

$A, C$ are independent; $B, C$ are independent;
$A \cap B, C$ are not independent. ($Pr[A \cap B \cap C] = 0 \neq Pr[A \cap B]Pr[C].$)
Pairwise Independence

Flip two fair coins. Let

- $A = \text{‘first coin is H’} = \{HT, HH\}$;
- $B = \text{‘second coin is H’} = \{TH, HH\}$;
- $C = \text{‘the two coins are different’} = \{TH, HT\}$.

$A, C$ are independent; $B, C$ are independent; $A \cap B, C$ are not independent. ($Pr[A \cap B \cap C] = 0 \neq Pr[A \cap B]Pr[C]$.)

If $A$ did not say anything about $C$ and $B$ did not say anything about $C$, then $A \cap B$ would not say anything about $C$. 
Example 2

Flip a fair coin 5 times.
Example 2

Flip a fair coin 5 times. Let $A_n = \text{`coin } n \text{ is H'}, \text{ for } n = 1, \ldots, 5$. 
Example 2

Flip a fair coin 5 times. Let $A_n = \text{\textquotesingle} \text{coin } n \text{ is H}\text{\textquotesingle}$, for $n = 1, \ldots, 5$. Then,

$$A_m, A_n \text{ are independent for all } m \neq n.$$
Example 2

Flip a fair coin 5 times. Let $A_n = \text{`coin } n \text{ is H'},$ for $n = 1, \ldots, 5.$

Then,

$$A_m, A_n \text{ are independent for all } m \neq n.$$ 

Also,

$$A_1 \text{ and } A_3 \cap A_5 \text{ are independent.}$$
Example 2

Flip a fair coin 5 times. Let $A_n = \text{‘coin } n \text{ is H’}$, for $n = 1, \ldots, 5$.

Then,

$$A_m, A_n \text{ are independent for all } m \neq n.$$  

Also,

$$A_1 \text{ and } A_3 \cap A_5 \text{ are independent.}$$

Indeed,

$$Pr[A_1 \cap (A_3 \cap A_5)] = \frac{1}{8} = Pr[A_1]Pr[A_3 \cap A_5].$$
Example 2

Flip a fair coin 5 times. Let $A_n = \text{`coin n is H'},$ for $n = 1, \ldots, 5.$

Then,

$$A_m, A_n \text{ are independent for all } m \neq n.$$ 

Also,

$$A_1 \text{ and } A_3 \cap A_5 \text{ are independent.}$$

Indeed,

$$Pr[A_1 \cap (A_3 \cap A_5)] = \frac{1}{8} = Pr[A_1] Pr[A_3 \cap A_5].$$

Similarly,

$$A_1 \cap A_2 \text{ and } A_3 \cap A_4 \cap A_5 \text{ are independent.}$$
Example 2

Flip a fair coin 5 times. Let $A_n = \text{`coin } n \text{ is H'},$ for $n = 1, \ldots, 5.$ Then,

$$A_m, A_n \text{ are independent for all } m \neq n.$$ 

Also,

$$A_1 \text{ and } A_3 \cap A_5 \text{ are independent.}$$

Indeed,

$$Pr[A_1 \cap (A_3 \cap A_5)] = \frac{1}{8} = Pr[A_1] Pr[A_3 \cap A_5].$$

Similarly,

$$A_1 \cap A_2 \text{ and } A_3 \cap A_4 \cap A_5 \text{ are independent.}$$

This leads to a definition ....
**Definition** Mutual Independence

(a) The events \( A_1, \ldots, A_5 \) are mutually independent if
\[
\Pr\left[ \bigcap_{k \in K} A_k \right] = \prod_{k \in K} \Pr[A_k],
\]
for all \( K \subseteq \{1, \ldots, 5\} \).

(b) More generally, the events \( \{A_j, j \in J\} \) are mutually independent if
\[
\Pr\left[ \bigcap_{k \in K} A_k \right] = \prod_{k \in K} \Pr[A_k],
\]
for all finite \( K \subseteq J \).

Example: Flip a fair coin forever. Let \( A_n = \text{`coin } n\text{ is H.'} \) Then the events \( A_n \) are mutually independent.
**Definition** Mutual Independence

(a) The events $A_1, \ldots, A_5$ are mutually independent if

\[
\Pr\left[ \bigcap_{k \in K} A_k \right] = \prod_{k \in K} \Pr[A_k],
\]

for all $K \subseteq \{1, \ldots, 5\}$.

(b) More generally, the events \{\(A_j, j \in J\}\} are mutually independent if

\[
\Pr\left[ \bigcap_{k \in K} A_k \right] = \prod_{k \in K} \Pr[A_k],
\]

for all finite $K \subseteq J$.

Example: Flip a fair coin forever. Let $A_n = 'coin \; n \; is \; H.'$ Then the events $A_n$ are mutually independent.
Definition Mutual Independence

(a) The events $A_1, \ldots, A_5$ are mutually independent if

$$Pr[\cap_{k \in K} A_k] = \prod_{k \in K} Pr[A_k], \text{ for all } K \subseteq \{1, \ldots, 5\}.$$
Mutual Independence

**Definition** Mutual Independence

(a) The events $A_1, \ldots, A_5$ are **mutually independent** if

$$Pr[\cap_{k \in K} A_k] = \prod_{k \in K} Pr[A_k], \text{ for all } K \subseteq \{1, \ldots, 5\}.$$  

(b) More generally, the events $\{A_j, j \in J\}$ are **mutually independent** if
Mutual Independence

**Definition** Mutual Independence

(a) The events $A_1, \ldots, A_5$ are **mutually independent** if

$$Pr[\bigcap_{k \in K} A_k] = \prod_{k \in K} Pr[A_k], \text{ for all } K \subseteq \{1, \ldots, 5\}.$$ 

(b) More generally, the events $\{A_j, j \in J\}$ are **mutually independent** if

$$Pr[\bigcap_{k \in K} A_k] = \prod_{k \in K} Pr[A_k], \text{ for all finite } K \subseteq J.$$
Definition Mutual Independence

(a) The events $A_1, \ldots, A_5$ are mutually independent if

$$Pr[\cap_{k \in K} A_k] = \prod_{k \in K} Pr[A_k], \text{ for all } K \subseteq \{1, \ldots, 5\}.$$ 

(b) More generally, the events $\{A_j, j \in J\}$ are mutually independent if

$$Pr[\cap_{k \in K} A_k] = \prod_{k \in K} Pr[A_k], \text{ for all finite } K \subseteq J.$$ 

Example: Flip a fair coin forever. Let $A_n = \text{‘coin } n \text{ is H.’}$ Then the events $A_n$ are mutually independent.
Mutual Independence

Theorem

(a) If the events \( \{A_j \mid j \in J \} \) are mutually independent and if \( K_1 \) and \( K_2 \) are disjoint finite subsets of \( J \), then \( \cap_{k \in K_1} A_k \) and \( \cap_{k \in K_2} A_k \) are independent.

(b) More generally, if the \( K_n \) are pairwise disjoint finite subsets of \( J \), then the events \( \cap_{k \in K_n} A_k \) are mutually independent.

(c) Also, the same is true if we replace some of the \( A_k \) by \( \bar{A}_k \).

Proof: See Notes 25, 2.7.
Theorem

(a) If the events \( \{A_j, j \in J\} \) are mutually independent and if \( K_1 \) and \( K_2 \) are disjoint finite subsets of \( J \), then
Mutual Independence

Theorem

(a) If the events \( \{ A_j, j \in J \} \) are mutually independent and if \( K_1 \) and \( K_2 \) are disjoint finite subsets of \( J \), then

\[
\cap_{k \in K_1} A_k \text{ and } \cap_{k \in K_2} A_k \text{ are independent.}
\]
Theorem

(a) If the events $\{A_j, j \in J\}$ are mutually independent and if $K_1$ and $K_2$ are disjoint finite subsets of $J$, then

$$\cap_{k \in K_1} A_k \text{ and } \cap_{k \in K_2} A_k \text{ are independent.}$$

(b) More generally, if the $K_n$ are pairwise disjoint finite subsets of $J$, then the events

$$\cap_{k \in K_n} A_k \text{ are mutually independent.}$$

Proof: See Notes 25, 2.7.
Mutual Independence

Theorem

(a) If the events \( \{ A_j, j \in J \} \) are mutually independent and if \( K_1 \) and \( K_2 \) are disjoint finite subsets of \( J \), then

\[ \bigcap_{k \in K_1} A_k \text{ and } \bigcap_{k \in K_2} A_k \] are independent.

(b) More generally, if the \( K_n \) are pairwise disjoint finite subsets of \( J \), then the events

\[ \bigcap_{k \in K_n} A_k \] are mutually independent.

(c) Also, the same is true if we replace some of the \( A_k \) by \( \bar{A}_k \).
Mutual Independence

Theorem

(a) If the events \(\{A_j, j \in J\}\) are mutually independent and if \(K_1\) and \(K_2\) are disjoint finite subsets of \(J\), then

\[\bigcap_{k \in K_1} A_k \text{ and } \bigcap_{k \in K_2} A_k\]

are independent.

(b) More generally, if the \(K_n\) are pairwise disjoint finite subsets of \(J\), then the events

\[\bigcap_{k \in K_n} A_k\]

are mutually independent.

(c) Also, the same is true if we replace some of the \(A_k\) by \(\bar{A}_k\).

Proof:
See Notes 25, 2.7.