Back to work...with some review.

Probability Space: $\Omega$, $Pr : \Omega \rightarrow [0, 1]$, $\sum_{\omega \in \Omega} Pr(\omega) = 1$. 
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$X \sim U[1, \ldots, n]$ $E[X] = \frac{n+1}{2}$, $Var(X) = \frac{n^2-1}{12}$.
Markov.

Markov:

For increasing function $f(x) \rightarrow \mathbb{R}^+$, $\Pr[X \geq a] \leq E[f(X)] f(a)$.

Proof: Take $f(x) = x$ in Markov.

Proof of Markov: Use random variable $Y = f(X)$ in Simple Markov.

Circular!

Proof of Simple Markov:

$E[X] = \sum x x \Pr[X = x] \geq \sum x \geq a x \Pr[X = x] \geq a \sum x \geq a x \Pr[X = x] = a \sum x \geq a \Pr[X \geq a]$. 

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\[\square\]
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Chebyshev’s Inequality

This is Pafnuty’s inequality:

\[ \Pr \left( |X - E[X]| > a \right) \leq \frac{\text{var}[X]}{a^2}, \text{for all } a > 0. \]
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Yes!
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Yes! The variance does measure the “deviations from the mean.”
Chebyshev and Poisson

Let $X = P(\lambda)$. Then, $E[X] = \lambda$ and $\text{var}[X] = \lambda$. Thus,

$$\Pr[|X - \lambda| \geq n] \leq \frac{\text{var}[X]}{n^2} = \frac{\lambda}{n^2}.$$
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By Markov's inequality, 

$$\Pr[X \geq a] \leq \frac{E[X^2]}{a^2} = \lambda + \lambda^2 a^2.$$ 

Also, if $a > \lambda$, then $X \geq a \Rightarrow X - \lambda \geq a - \lambda > 0 \Rightarrow |X - \lambda| \geq a - \lambda$. 

Hence, for $a > \lambda$, 

$$\Pr[X \geq a] \leq \Pr[|X - \lambda| \geq a - \lambda] \leq \lambda(\lambda - a).$$
Chebyshev and Poisson (continued)

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Let $X = P(\lambda)$. Then, $E[X] = \lambda$ and $\text{var}[X] = \lambda$. By Markov's inequality,

$$Pr[X \geq a] \leq \frac{E[X^2]}{a^2} = \frac{\lambda + \lambda^2}{a^2}.$$ 

Also, if $a > \lambda$, then $X \geq a \Rightarrow X - \lambda \geq a - \lambda > 0 \Rightarrow |X - \lambda| \geq a - \lambda$.

Hence, for $a > \lambda$, $Pr[X \geq a] \leq Pr[|X - \lambda| \geq a - \lambda] \leq \frac{\lambda}{(a - \lambda)^2}$. 

Chebyshev and Poisson (continued)

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Hence, for $a > \lambda$, $Pr[X \geq a] \leq Pr[|X - \lambda| \geq a - \lambda] \leq \frac{\lambda}{(a-\lambda)^2}.\tag{2}$

[Graph showing Chebyshev and Markov inequalities with $X = P(\lambda), \lambda = 10$. The graph illustrates the comparison between the two inequalities with $g(x) = x^2$. The Chebyshev inequality is shown with a yellow line, while the Markov inequality is shown with a blue line. The graph includes points at $a = 14, 16, 18, 20, 22, 24$. The $y$-axis ranges from 0.0 to 0.7, and the $x$-axis ranges from 14 to 24.]
Fraction of $H$’s

Here is a classical application of Chebyshev’s inequality.

\[
\text{Let } X_m = 1 \text{ if the } m\text{-th flip of a fair coin is } H \text{ and } X_m = 0 \text{ otherwise.}
\]

Define
\[
Y_n = X_1 + \cdots + X_n,
\]
for \( n \geq 1 \).

We want to estimate
\[
\Pr \left[ \left| Y_n - \frac{1}{2} \right| \geq 0.1 \right] = \Pr \left[ Y_n \leq 0.4 \text{ or } Y_n \geq 0.6 \right].
\]
By Chebyshev,
\[
\Pr \left[ \left| Y_n - \frac{1}{2} \right| \geq 0.1 \right] \leq \frac{\text{var} \left[ Y_n \right]}{0.1^2} = 100 \text{var} \left[ Y_n \right].
\]
Now,
\[
\text{var} \left[ Y_n \right] = \frac{1}{n^2} \left( \text{var} \left[ X_1 \right] + \cdots + \text{var} \left[ X_n \right] \right) \leq \frac{1}{4n}.
\]
Var \( (X_i) = p (1 - lp) \leq (0.5)(0.5) = 0.25 \)
Fraction of \( H \)'s

Here is a classical application of Chebyshev’s inequality. How likely is it that the fraction of \( H \)'s differs from 50%?

\[
\text{Let } X_m = \begin{cases} 
1 & \text{if the } m\text{-th flip of a fair coin is } H \\
0 & \text{otherwise}
\end{cases}
\]

Define \( Y_n = X_1 + \cdots + X_n \), for \( n \geq 1 \).

We want to estimate \( \Pr[|Y_n - 0.5| \geq 0.1] = \Pr[Y_n \leq 0.4 \text{ or } Y_n \geq 0.6] \).

By Chebyshev, \( \Pr[|Y_n - 0.5| \geq 0.1] \leq \frac{\text{var}[Y_n]}{(0.1)^2} = 100 \text{ var}[Y_n] \).

Now, \( \text{var}[Y_n] = \frac{1}{n^2} (\text{var}[X_1] + \cdots + \text{var}[X_n]) = \frac{1}{n^2} \cdot \frac{1}{4} = \frac{1}{4n} \).

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For $n = 1,000$, we find that this probability is less than 2.5%.
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We look at a calculation of this next.
Weak Law of Large Numbers

**Theorem** Weak Law of Large Numbers

Let $X_1, X_2, \ldots$ be pairwise independent with the same distribution and mean $\mu$. Then, for all $\varepsilon > 0$, $\Pr[|X_1 + \cdots + X_n - n\mu| \geq \varepsilon] \to 0$, as $n \to \infty$.

**Proof:**

Let $Y_n = X_1 + \cdots + X_n$. Then

$$\Pr[|Y_n - n\mu| \geq \varepsilon] \leq \text{var}(Y_n) \frac{\varepsilon^2}{\text{var}(X_1 + \cdots + X_n)} = \frac{n\text{var}(X_1)}{n\varepsilon^2} \to 0,$$

as $n \to \infty$. 


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WLLN: \( X_m \) i.i.d. \( \Rightarrow \frac{X_1 + \cdots + X_n}{n} \approx E[X] \)
Probability: Midterm 2 Review.

- Framework:
  - Probability Space
  - Conditional Probability & Bayes’ Rule
  - Independence
  - Mutual Independence
Review: Probability Space

- Sample Space
- $\Omega$
- $\omega_1$, $\omega_2$
- Samples (Outcomes)

- $0 \leq \Pr[\omega] \leq 1$
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Review: Probability Space

Sample Space

$\Omega$

$\omega_1, \omega_2$

Samples (Outcomes)

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Pr[$A|B$] = Pr[$A \cap B$] / Pr[$B$].

Pr[$A \cap B \cap C$] = Pr[$A$] * Pr[$B|A$] * Pr[$C|A \cap B$].
Review: Probability Space

Sample Space

$\Omega$ 

$\omega_1, \omega_2, \ldots$ 

Samples (Outcomes)

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Review: Probability Space

Sample Space

\[ \Omega \]

Samples (Outcomes)

Fraction \( p_1 \) of circumference

\[ \sum_{\omega} Pr[\omega] = 1 \]

\[ 0 \leq Pr[\omega] \leq 1 \]

Pr[\( A \mid B \)] = \( \frac{Pr[A \cap B]}{Pr[B]} \).

\[ Pr[A \cap B \cap C] = Pr[A]Pr[B \mid A]Pr[C \mid A \cap B]. \]
Review: Bayes’ Rule
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Review: Bayes’ Rule

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- **Conditional Probabilities:** $Pr[B|A_n] = q_n, n = 1, \ldots, N$
- $\Rightarrow$ **Posteriors:** $Pr[A_n|B] = \frac{p_n q_n}{p_1 q_1 + \cdots + p_N q_N}$
Bayes’ Rule: Examples

Let $p'_n = \Pr[A_n | B]$ be the posterior probabilities. Thus, $p'_n = \frac{p_n q_n}{p_1 q_1 + \cdots + p_N q_n}$.

Questions:

▶ if $q_n > q_k$, then $p'_n > p'_k$? Not necessarily.
▶ if $p_n > p_k$, then $p'_n > p'_k$? Not necessarily.
▶ if $p_n > p_k$ and $q_n > q_k$, then $p'_n > p'_k$? Yes.
▶ if $q_n = 1$, then $p'_n > 0$? Not necessarily.
▶ if $p_n = \frac{1}{N}$ for all $n$, then MLE = MAP? Yes.
Let $p'_n = Pr[A_n|B]$ be the posterior probabilities.
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\end{itemize}
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Review: Independence

"First coin yields 1" and "Sum is 7" are pairwise, but not mutually independent.

If \( \{A_j, i \in J\} \) are mutually independent, then \( A_1 \cap \overline{A_2} \) and \( A_3 \setminus A_4 \) are independent.

Our intuitive meaning of "independent events" is mutual independence.
Review: Independence

“First coin yields 1” and ”Sum is 7” are independent
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"First coin yields 1" and "Sum is 7" are independent Pairwise, but not mutually

Our intuitive meaning of "independent events" is mutual independence.

\[ \begin{align*} &\{A_j, i \in J\} \\
&\left[ A_1 \cap \bar{A}_2 \right] \Delta A_3 \\
&\text{and } A_4 \setminus A_5 \end{align*} \]

\( \Omega = \{1, \ldots, 6\}^2 \)

\( A = \{(1, 6), \ldots, (6,1)\} \)

\( B = \{(1,1), \ldots, (1,6)\} \)

A = ‘sum is 7’
Review: Independence

“First coin yields 1” and ”Sum is 7” are independent

Pairwise, but not mutually

If \( \{A_j, i \in J\} \) are mutually independent, then \([A_1 \cap \bar{A}_2] \Delta A_3\) and \(A_4 \setminus A_5\) are independent.

Our intuitive meaning of “independent events” is mutual independence.
Review: Independence

A and B are independent if \( \Pr[A \cap B] = \Pr[A] \Pr[B] \).

\( \{A_j, j \in J\} \) are mutually independent if \( \Pr[\bigcap_{j \in K} A_j] = \prod_{j \in K} \Pr[A_j] \), for all finite \( K \subset J \).

Thus, \( A, B, C, D \) are mutually independent if there are independent 2 by 2:

\( \Pr[A \cap B] = \Pr[A] \Pr[B] \),...,

\( \Pr[C \cap D] = \Pr[C] \Pr[D] \)

by 3:

\( \Pr[A \cap B \cap C] = \Pr[A] \Pr[B] \Pr[C] \),...,

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- by 4: $\Pr \left[ A \cap B \cap C \cap D \right] = \Pr \left[ A \right] \Pr \left[ B \right] \Pr \left[ C \right] \Pr \left[ D \right]$.
Consider the uniform probability space and the events $A$, $B$, $C$, $D$. Which maximal collections of events among $A$, $B$, $C$, $D$ are pairwise independent?

\{
A, B, C\}, \{B, C, D\}

Can you find three events among $A$, $B$, $C$, $D$ that are mutually independent? No: We would need an outcome with probability $1/8$. 
Consider the uniform probability space and the events $A, B, C, D$.

Which maximal collections of events among $A, B, C, D$ are pairwise independent?

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Review: Collisions & Collecting

Collisions:

\[ Pr[\text{no collision}] \approx e^{-\frac{m^2}{2n}} \]
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Collisions:

\[ Pr[\text{no collision}] \approx e^{-m^2/2n} \]

Collecting:

\[ Pr[\text{miss Wilson}] \approx e^{-m/n} \]

\[ Pr[\text{miss at least one}] \leq ne^{-m/n} \]
Approximations:

\[ \ln(1 - \epsilon) \approx -\epsilon \]
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Sums:

\[ (a + b)^n = \sum_{m=0}^{n} \binom{n}{m} a^m b^{n-m} \]
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Sums:

\[ (a + b)^n = \sum_{m=0}^{n} \binom{n}{m} a^m b^{n-m} \]
\[ 1 + 2 + \cdots + n = \frac{n(n+1)}{2}; \]
Math Tricks, continued

Symmetry:

E.g., if we pick balls from a bag, with no replacement,
\[ \Pr[\text{ball 5 is red}] = \Pr[\text{ball 1 is red}] \]
Order of balls = permutation. All permutations have same probability.

Union Bound:
\[ \Pr[A \cup B \cup C] \leq \Pr[A] + \Pr[B] + \Pr[C] \]

Inclusion/Exclusion:
\[ \Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B] \]

Total Probability:
\[ \Pr[B] = \Pr[A_1] \Pr[B|A_1] + \cdots + \Pr[A_n] \Pr[B|A_n] \]

An L_2-bounded martingale converges almost surely. Just kidding!
Math Tricks, continued

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An \( L^2 \)-bounded martingale converges almost surely. Just kidding!
A mini-quizz

True or False:

- \( Pr[A \cup B] = Pr[A] + Pr[B] \).

False

True iff disjoint.

False

True iff independent.

False

\( A \cap B = \emptyset \Rightarrow A, B \) independent.

False

\( \Pr[A \cap B \cap C] = \Pr[A] \Pr[B | A] \Pr[C | B] \).

False
A mini-quizz

True or False:

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A mini-quizz

True or False:

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A mini-quizz

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A mini-quizz

True or False:

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True or False:

3. $A \cap B = \emptyset \Rightarrow A, B$ independent.
A mini-quizz

True or False:

- $A \cap B = \emptyset \Rightarrow A, B$ independent. False
A mini-quizz

True or False:

- $Pr[A \cap B] = Pr[A]Pr[B]$. **False** True iff independent.
- $A \cap B = \emptyset \Rightarrow A, B$ independent. **False**
- For all $A, B$, one has $Pr[A|B] \geq Pr[A]$. **False**
A mini-quizz

True or False:

3. $A \cap B = \emptyset \Rightarrow A, B$ independent. False
4. For all $A, B$, one has $Pr[A|B] \geq Pr[A]$. False
True or False:

- $\Pr[A \cup B] = \Pr[A] + \Pr[B]$. False True iff disjoint.
- $\Pr[A \cap B] = \Pr[A] \Pr[B]$. False True iff independent.
- $A \cap B = \emptyset \Rightarrow A, B$ independent. False
- For all $A, B$, one has $\Pr[A|B] \geq \Pr[A]$. False
- $\Pr[A \cap B \cap C] = \Pr[A] \Pr[B|A] \Pr[C|B]$. 

A mini-quizz
True or False:

- $A \cap B = \emptyset \Rightarrow A, B$ independent. False
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- $Pr[A \cap B \cap C] = Pr[A]Pr[B|A]Pr[C|B]$. False
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- \( Pr[A \cap B \cap C] = Pr[A] Pr[B|A] Pr[C|B] \). False
A mini-quizz; part 2

- $\Omega = \{1, 2, 3, 4\}$, uniform.

- $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{1, 4\}$. $A$, $B$, $C$ pairwise independent.

- Is it true that $(A \cap B)$ and $C$ are independent? No.

- Assume $\Pr[C | A] > \Pr[C | B]$. Is it true that $\Pr[A | C] > \Pr[B | C]$? No.

- Deal two cards from a 52-card deck. What is the probability that the value of the first card is strictly larger than that of the second?

  - $\Pr[\text{same}] = \frac{3}{51}$.
  - $\Pr[\text{different}] = \frac{48}{51}$.
  - $\Pr[\text{first} > \text{second}] = \frac{24}{51}$. 

- Find events $A$, $B$, $C$ that are pairwise independent, not mutually.

- $\Omega = \{1, 2, 3, 4\}$, uniform.
A mini-quizz; part 2

- \( \Omega = \{1, 2, 3, 4\} \), uniform. Find events \( A, B, C \) that are pairwise independent, not mutually.

- \( A = \{1, 2\} \), \( B = \{1, 3\} \), \( C = \{1, 4\} \).

- \( A \), \( B \), \( C \) pairwise independent.

- Is it true that \( (A \cap B) \cap C \) are independent? No.

- In the example above, \( \Pr[A \cap B \cap C] \neq \Pr[A \cap B] \Pr[C] \).

- Assume \( \Pr[C | A] > \Pr[C | B] \).

- Is it true that \( \Pr[A | C] > \Pr[B | C] \)? No.

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- $A, B, C$ pairwise independent.
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  \[ A = \{1, 2\}, B = \{1, 3\}, C = \{1, 4\}. \]

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A mini-quizz; part 2

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  $$A = \{1, 2\}, B = \{1, 3\}, C = \{1, 4\}.$$  

- $A, B, C$ pairwise independent. Is it true that $(A \cap B)$ and $C$ are independent?
  
  No. In example above, $Pr[A \cap B \cap C] \neq Pr[A \cap B]Pr[C]$.
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- Assume $Pr[C|A] > Pr[C|B]$.  

- Deal two cards from a 52-card deck. What is the probability that the value of the first card is strictly larger than that of the second?
  
  $Pr[same] = \frac{3}{51}$.
  
  $Pr[different] = \frac{48}{51}$.
  
  $Pr[first > second] = \frac{24}{51}$. 
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Discrete Math: Review
Modular Arithmetic Inverses and GCD

$x$ has inverse modulo $m$ if and only if $\text{gcd}(x, m) = 1$. 
Modular Arithmetic Inverses and GCD

\(x\) has inverse modulo \(m\) if and only if \(gcd(x, m) = 1\).

Group structures more generally.
Modular Arithmetic Inverses and GCD

\[ x \text{ has inverse modulo } m \text{ if and only if } \gcd(x, m) = 1. \]

Group structures more generally.

Proof Idea:
\{0x, \ldots, (m-1)x\} are distinct modulo \( m \) if and only if \( \gcd(x, m) = 1. \)
Modular Arithmetic Inverses and GCD

$x$ has inverse modulo $m$ if and only if $gcd(x, m) = 1$.

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$gcd(x, y) = gcd(y, x - y)$
Modular Arithmetic Inverses and GCD

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Finding gcd.

$\gcd(x, y) = \gcd(y, x - y) = \gcd(y, x \pmod{y})$. 

Idea: egcd.

$\gcd$ produces 1 by adding and subtracting multiples of $x$ and $y$. 

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Give recursive Algorithm!
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Extended-gcd($x, y$)
Modular Arithmetic Inverses and GCD

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Extended-gcd\((x, y)\) returns \((d, a, b)\)
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egcd$(x, m) = (1, a, b)$

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Choose $e = 7$, since $\gcd(7, 60) = 1.$
Example: \( p = 7, \ q = 11. \)

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\[ (p - 1)(q - 1) = 60 \]

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7(0) + 60(1) = 60
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\begin{align*}
7(0) + 60(1) &= 60 \\
7(1) + 60(0) &= 7 \\
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\end{align*}
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Confirm:
Example: $p = 7$, $q = 11$.

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Confirm: $-119 + 120 = 1$
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Confirm: \( -119 + 120 = 1 \)

\( d = e^{-1} = -17 = 43 = \ (\text{mod} \ 60) \)
Fermat from Bijection.

Fermat’s Little Theorem: For prime $p$, and $a \not\equiv 0 \pmod{p}$,
Fermat from Bijection.

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**Fermat’s Little Theorem:** For prime $p$, and $a \not\equiv 0 \pmod{p}$,
\[ a^{p-1} \equiv 1 \pmod{p}. \]

**Proof:** Consider $T = \{a \cdot 1 \pmod{p}, \ldots, a \cdot (p - 1) \pmod{p}\}$. 

Since multiplication is commutative, the product of elements in $T$ is the same as the product of elements in $\{1, \ldots, p-1\}$ modulo $p$. Each of $2, \ldots, (p-1)$ has an inverse modulo $p$, multiply by inverses to get $a^{p-1} \equiv 1 \pmod{p}$. 


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$T$ is range of function $f(x) = ax \pmod{p}$ for set $S = \{1, \ldots, p-1\}$.  

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Invertible function:
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$T \subseteq S$ since $0 \not\in T$. 
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Invertible function: one-to-one.

- $T \subseteq S$ since $0 \not\in T$.
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Product of elts of $T = \text{Product of elts of } S$. 

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RSA

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\[ N = p, q \]
with \( \gcd(e, (p-1)(q-1)) = 1 \).

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Theorem:

\[ x^{ed} = x \pmod{N} \]

Proof:

\[ x^{ed} - x \] is divisible by \( p \) and \( q \) \( \Rightarrow \) theorem!

\[ x^{ed} - x = x^k(p-1)(q-1) + 1 - x = x^{((x^k(q-1))(p-1))} - 1 \]

If \( x \) is divisible by \( p \), the product is.
Otherwise \( (x^k(q-1))^{p-1} = 1 \pmod{p} \) by Fermat.

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Polynomials

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Proof Idea:
Any polynomial with roots $r_1, \ldots, r_k$.  

**Property 2:** There is exactly 1 polynomial of degree $\leq d$ with arithmetic modulo prime $p$ that contains any $d+1$: 
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Proof Ideas:
Lagrange Interpolation gives existence.
Property 1 gives uniqueness.
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Secret Sharing: $k$ out of $n$ people know secret.
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  Scheme: degree $n - 1$ polynomial, $P(x)$. 
Applications.

**Property 2:** There is exactly 1 polynomial of degree $\leq d$ with arithmetic modulo prime $p$ that contains any $d + 1$ points: $(x_1, y_1), \ldots, (x_{d+1}, y_{d+1})$ with $x_i$ distinct.

Secret Sharing: $k$ out of $n$ people know secret.
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   **Secret:** $P(0)$ **Shares:** $(1, P(1)), \ldots (n, P(n))$. 
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Erasure Coding: $n$ packets, $k$ losses.
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Corruptions Coding: $n$ packets, $k$ corruptions.
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- **Send:** \((0, P(0)), \ldots (n+2k-1, P(n+2k-1))\).
- **Recovery:** \( P(x) \) is only consistent polynomial with \( n + k \) points.
Applications.

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- Send: $(0, P(0)), \ldots (n+2k-1, P(n+2k-1))$.
- Recovery: $P(x)$ is only consistent polynomial with $n+k$ points.
  Property 2 and pigeonhole principle.
Idea: Error locator polynomial of degree $k$ with zeros at errors.
Welsh-Berlekamp

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For all points $i = 1, \ldots, i, n + 2k$, $P(i)E(i) = R(i)E(i) \pmod{p}$
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For all points $i = 1, \ldots, i, n + 2k$, $P(i)E(i) = R(i)E(i) \pmod{p}$

since $E(i) = 0$ at points where there are errors.
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Let $Q(x) = P(x)E(x)$. 
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For all points $i = 1, \ldots, i, n+2k$, $P(i)E(i) = R(i)E(i)$ (mod $p$)
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Let $Q(x) = P(x)E(x)$.

$$Q(x) = a_{n+k-1}x^{n+k-1} + \cdots a_0.$$
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Gives system of $n + 2k$ linear equations.
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$$a_{n+k-1} + \cdots + a_0 \equiv R(1)(1 + b_{k-1} \cdots b_0) \pmod{p}$$
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$$\vdots$$
Welsh-Berlekamp

Idea: Error locator polynomial of degree \( k \) with zeros at errors.

For all points \( i = 1, \ldots, i, n+2k \), \( P(i)E(i) = R(i)E(i) \) \((\text{mod } p)\)

since \( E(i) = 0 \) at points where there are errors.

Let \( Q(x) = P(x)E(x) \).

\[
Q(x) = a_{n+k-1}x^{n+k-1} + \cdots a_0. \\
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\]

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a_{n+k-1}(2)^{n+k-1} + \cdots a_0 \equiv R(2)((2)^k + b_{k-1}(2)^{k-1} \cdots b_0) \pmod{p} \\
\vdots \\
a_{n+k-1}(m)^{n+k-1} + \cdots a_0 \equiv R(m)((m)^k + b_{k-1}(m)^{k-1} \cdots b_0) \pmod{p}
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\[
\begin{align*}
    a_{n+k-1} + \cdots a_0 &\equiv R(1)(1 + b_{k-1} \cdots b_0) \pmod{p} \\
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    & \vdots \\
    a_{n+k-1}(m)^{n+k-1} + \cdots a_0 &\equiv R(m)((m)^k + b_{k-1}(m)^{k-1} \cdots b_0) \pmod{p}
\end{align*}
\]

..and $n + 2k$ unknown coefficients of $Q(x)$ and $E(x)$!
Welsh-Berlekamp

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Solve for coefficients of $Q(x)$ and $E(x)$. 
Welsh-Berlekamp

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Solve for coefficients of $Q(x)$ and $E(x)$.

Find $P(x) = Q(x)/E(x)$. 
Welsh-Berlekamp

Idea: Error locator polynomial of degree \( k \) with zeros at errors.

For all points \( i = 1, \ldots, i, n+2k \), \( P(i)E(i) = R(i)E(i) \pmod{p} \)

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Gives system of \( n+2k \) linear equations.

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\[\vdots\]

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\]

..and \( n+2k \) unknown coefficients of \( Q(x) \) and \( E(x) \)!

Solve for coefficients of \( Q(x) \) and \( E(x) \).

Find \( P(x) = Q(x)/E(x) \).
Welsh-Berlekamp

Idea: Error locator polynomial of degree \( k \) with zeros at errors.

For all points \( i = 1, \ldots, i, n + 2k \), \( P(i)E(i) = R(i)E(i) \) (mod \( p \))

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\]
\[
a_{n+k-1}(2)^{n+k-1} + \cdots + a_0 \equiv R(2)((2)^k + b_{k-1}(2)^{k-1} \cdots b_0) \pmod{p}
\]
\[
\vdots
\]
\[
a_{n+k-1}(m)^{n+k-1} + \cdots + a_0 \equiv R(m)((m)^k + b_{k-1}(m)^{k-1} \cdots b_0) \pmod{p}
\]

..and \( n + 2k \) unknown coefficients of \( Q(x) \) and \( E(x) \)!

Solve for coefficients of \( Q(x) \) and \( E(x) \).

Find \( P(x) = Q(x)/E(x) \).
Welsh-Berlekamp

Idea: Error locator polynomial of degree $k$ with zeros at errors.

For all points $i = 1, \ldots, i, n+2k$, $P(i)E(i) = R(i)E(i) \pmod{p}$ since $E(i) = 0$ at points where there are errors.

Let $Q(x) = P(x)E(x)$.

\[ Q(x) = a_{n+k-1}x^{n+k-1} + \cdots + a_0. \]
\[ E(x) = x^k + b_{k-1}x^{k-1} + \cdots + b_0. \]

Gives system of $n+2k$ linear equations.

\[ a_{n+k-1} + \cdots + a_0 \equiv R(1)(1 + b_{k-1} \cdots b_0) \pmod{p} \]
\[ a_{n+k-1}(2)^{n+k-1} + \cdots + a_0 \equiv R(2)((2)^k + b_{k-1}(2)^{k-1} \cdots b_0) \pmod{p} \]
\[ \vdots \]
\[ a_{n+k-1}(m)^{n+k-1} + \cdots + a_0 \equiv R(m)((m)^k + b_{k-1}(m)^{k-1} \cdots b_0) \pmod{p} \]

..and $n+2k$ unknown coefficients of $Q(x)$ and $E(x)$!

Solve for coefficients of $Q(x)$ and $E(x)$.

\[ \text{Find } P(x) = Q(x)/E(x). \]
Counting

First Rule
Counting

First Rule
Second Rule
Counting

First Rule
Second Rule
Stars/Bars
Counting

First Rule
Second Rule
Stars/Bars
Common Scenarios: Sampling, Balls in Bins.
Counting

First Rule
Second Rule
Stars/Bars
Common Scenarios: Sampling, Balls in Bins.
Sum Rule. Inclusion/Exclusion.
First Rule
Second Rule
Stars/Bars
Common Scenarios: Sampling, Balls in Bins.
Sum Rule. Inclusion/Exclusion.
Combinatorial Proofs.
Counting

First Rule
Second Rule
Stars/Bars
Common Scenarios: Sampling, Balls in Bins.
Sum Rule. Inclusion/Exclusion.
Combinatorial Proofs.
Example: visualize.

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide...when possible.
Example: visualize.

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide..when possible.

3 card Poker deals: 52
Example: visualize.

First rule: $n_1 \times n_2 \cdots \times n_3$. Product Rule.
Second rule: when order doesn’t matter divide..when possible.

3 card Poker deals: $52 \times 51$
Example: visualize.

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide...when possible.

3 card Poker deals: \( 52 \times 51 \times 50 \)
Example: visualize.

First rule: $n_1 \times n_2 \cdots \times n_3$. Product Rule.
Second rule: when order doesn’t matter divide when possible.

3 card Poker deals: $52 \times 51 \times 50 = \frac{52!}{49!}$. 
Example: visualize.

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide..when possible.

3 card Poker deals: \( 52 \times 51 \times 50 = \frac{52!}{49!} \). First rule.
Example: visualize.

**First rule:** \( n_1 \times n_2 \cdots \times n_3 \). **Product Rule.**
Second rule: when order doesn’t matter divide.. when possible.

3 card Poker deals: \( 52 \times 51 \times 50 = \frac{52!}{49!} \). First rule.
Poker hands: \( \Delta \)?
Example: visualize.

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide..when possible.

3 card Poker deals: \( 52 \times 51 \times 50 = \frac{52!}{49!} \). First rule.
Poker hands: \( \Delta? \)
Hand: \( Q, K, A. \)
Example: visualize.

First rule: \( n_1 \times n_2 \cdots \times n_3 \). **Product Rule.**
Second rule: when order doesn’t matter divide..when possible.

3 card Poker deals: \( 52 \times 51 \times 50 = \frac{52!}{49!} \). First rule.
Poker hands: \( \Delta ? \)
   Hand: \( Q, K, A \).
   Deals: \( Q, K, A \),
Example: visualize.

First rule: $n_1 \times n_2 \cdots \times n_3$. Product Rule.
Second rule: when order doesn’t matter divide..when possible.

3 card Poker deals: $52 \times 51 \times 50 = \frac{52!}{49!}$. First rule.
Poker hands: $\Delta$?
   Hand: Q, K, A.
   Deals: Q, K, A, Q, A, K,
Example: visualize.

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide..when possible.

\[ \Delta \]

3 card Poker deals: \( 52 \times 51 \times 50 = \frac{52!}{49!} \). First rule.
Poker hands: \( \Delta \)?
Hand: Q, K, A.
Example: visualize.

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide..when possible.

\[ \Delta \]

3 card Poker deals: \( 52 \times 51 \times 50 = \frac{52!}{49!} \). First rule.

Poker hands: \( \Delta \)?

Hand: \( Q, K, A \).
\( \Delta = 3 \times 2 \times 1 \)
Example: visualize.

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide when possible.

\[
\begin{array}{c}
\cdots \quad \Delta \\
\cdots \\
\cdots
\end{array}
\]

3 card Poker deals: \( 52 \times 51 \times 50 = \frac{52!}{49!} \). First rule.
Poker hands: \( \Delta \)?
   Hand: Q, K, A.
\( \Delta = 3 \times 2 \times 1 \) First rule again.
Example: visualize.

First rule: $n_1 \times n_2 \cdots \times n_3$. Product Rule.
Second rule: when order doesn’t matter divide...when possible.

\[
\Delta
\]

3 card Poker deals: $52 \times 51 \times 50 = \frac{52!}{49!}$. First rule.
Poker hands: $\Delta$?
  Hand: $Q, K, A$.

$\Delta = 3 \times 2 \times 1$ First rule again.
Total:
Example: visualize.

**First rule:** \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.

**Second rule:** when order doesn’t matter divide...when possible.

3 card Poker deals: \( 52 \times 51 \times 50 = \frac{52!}{49!} \). First rule.

Poker hands: \( \Delta \)?

**Hand:** Q, K, A.

**Deals:** Q, K, A, Q, A, K, K, A, Q, K, A, Q, A, K, Q, A, Q, K.

\( \Delta = 3 \times 2 \times 1 \) First rule again.

**Total:** \( \frac{52!}{49!3!} \)
Example: visualize.

**First rule:** \( n_1 \times n_2 \cdots \times n_3 \). **Product Rule.**

**Second rule:** when order doesn’t matter divide...when possible.

3 card Poker deals: \( 52 \times 51 \times 50 = \frac{52!}{49!} \). First rule.

Poker hands: \( \Delta \)?

- **Hand:** Q, K, A.
- **Deals:** Q, K, A, Q, A, K, K, A, Q, K, A, Q, A, K, Q, A, Q, K.

\( \Delta = 3 \times 2 \times 1 \) First rule again.

**Total:** \( \frac{52!}{49!3!} \) Second Rule!
Example: visualize.

First rule: $n_1 \times n_2 \cdots \times n_3$. Product Rule.
Second rule: when order doesn’t matter divide..when possible.

3 card Poker deals: $52 \times 51 \times 50 = \frac{52!}{49!}$. First rule.

Poker hands: $\Delta$?

Hand: $Q, K, A$.


$\Delta = 3 \times 2 \times 1$ First rule again.

Total: $\frac{52!}{49!3!}$ Second Rule!

Choose $k$ out of $n$. 
Example: visualize.

**First rule:** \( n_1 \times n_2 \cdots \times n_3 \). **Product Rule.**

**Second rule:** when order doesn’t matter divide..when possible.

3 card Poker deals: \( 52 \times 51 \times 50 = \frac{52!}{49!} \). First rule.

Poker hands: \( \Delta \)?
- **Hand**: Q, K, A.
- **Deals**: Q, K, A, Q, A, K, K, A, Q, K, A, Q, A, K, Q, A, Q, K.

\( \Delta = 3 \times 2 \times 1 \) First rule again.

**Total:** \( \frac{52!}{49!3!} \) Second Rule!

Choose \( k \) out of \( n \).
- Ordered set: \( \frac{n!}{(n-k)!} \)
Example: visualize.

First rule: $n_1 \times n_2 \cdots \times n_3$. Product Rule.
Second rule: when order doesn’t matter divide..when possible.

3 card Poker deals: $52 \times 51 \times 50 = \frac{52!}{49!}$. First rule.

Poker hands: $\Delta$?
   Hand: Q, K, A.

$\Delta = 3 \times 2 \times 1$ First rule again.
Total: $\frac{52!}{49!3!}$ Second Rule!

Choose $k$ out of $n$.
Ordered set: $\frac{n!}{(n-k)!}$

What is $\Delta$?
Example: visualize.

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide..when possible.

3 card Poker deals: \( 52 \times 51 \times 50 = \frac{52!}{49!} \). First rule.
Poker hands: \( \Delta \)?

Hand: Q, K, A.
\( \Delta = 3 \times 2 \times 1 \) First rule again.
Total: \( \frac{52!}{49!3!} \) Second Rule!

Choose \( k \) out of \( n \).
Ordered set: \( \frac{n!}{(n-k)!} \)
What is \( \Delta \)? \( k! \)
Example: visualize.

**First rule:** \( n_1 \times n_2 \cdots \times n_3 \). **Product Rule.**

**Second rule:** when order doesn’t matter divide...when possible.

3 card Poker deals: \( 52 \times 51 \times 50 = \frac{52!}{49!} \). First rule.

Poker hands: \( \Delta \)?

- **Hand:** Q, K, A.
- **Deals:** Q, K, A, Q, A, K, K, A, Q, K, A, Q, A, K, Q, A, Q, K.

\( \Delta = 3 \times 2 \times 1 \) First rule again.

**Total:** \( \frac{52!}{49!3!} \) **Second Rule!**

Choose \( k \) out of \( n \).

- **Ordered set:** \( \frac{n!}{(n-k)!} \)

What is \( \Delta \)? \( k! \) First rule again.
Example: visualize.

**First rule:** $n_1 \times n_2 \cdots \times n_3$. **Product Rule.**
**Second rule:** when order doesn’t matter divide..when possible.

3 card Poker deals: $52 \times 51 \times 50 = \frac{52!}{49!}$. First rule.

Poker hands: $\Delta$?

Hand: Q, K, A.


$\Delta = 3 \times 2 \times 1$ First rule again.

Total: $\frac{52!}{49!3!}$ Second Rule!

Choose $k$ out of $n$.

Ordered set: $\frac{n!}{(n-k)!}$

What is $\Delta$? $k!$ First rule again.

$\implies$ Total: $\frac{n!}{(n-k)!k!}$
Example: visualize.

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide...when possible.

3 card Poker deals: \( 52 \times 51 \times 50 = \frac{52!}{49!} \). First rule.
Poker hands: \( \Delta \)?

Hand: Q, K, A.
\( \Delta = 3 \times 2 \times 1 \) First rule again.
Total: \( \frac{52!}{49!3!} \) Second Rule!

Choose \( k \) out of \( n \).
Ordered set: \( \frac{n!}{(n-k)!} \)
What is \( \Delta \)? \( k! \) First rule again.
\( \Rightarrow \) Total: \( \frac{n!}{(n-k)!k!} \) Second rule.
Example: visualize

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide when possible.
**Example: visualize**

First rule: \( n_1 \times n_2 \cdots \times n_3 \). **Product Rule.**
Second rule: when order doesn’t matter divide..when possible.

... Of course if the order matters

Orderings of ANAGRAM?
Example: visualize

First rule: $n_1 \times n_2 \cdots \times n_3$. Product Rule.
Second rule: when order doesn’t matter divide..when possible.

Orderings of ANAGRAM?
Ordered Set: 7!
Example: visualize

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide..when possible.

... ... ...

Orderings of ANAGRAM?
Ordered Set: 7! First rule.
Example: visualize

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide..when possible.

Orderings of ANAGRAM?
Ordered Set: 7! First rule.
A’s are the same!
Example: visualize

First rule: $n_1 \times n_2 \cdots \times n_3$. Product Rule.
Second rule: when order doesn’t matter divide.. when possible.

Orderings of ANAGRAM?
Ordered Set: 7! First rule.
A’s are the same!
What is $\Delta$?
Example: visualize

First rule: $n_1 \times n_2 \cdots \times n_3$. Product Rule.
Second rule: when order doesn’t matter divide..when possible.

Orderings of ANAGRAM?
Ordered Set: 7! First rule.
A’s are the same!
What is $\Delta$?
ANAGRAM
Example: visualize

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide..when possible.

Orderings of ANAGRAM?
Ordered Set: 7! First rule.
A’s are the same!
What is \( \Delta \)?
ANAGRAM
\( A_1 NA_2 GRA_3 M \),
First rule: $n_1 \times n_2 \cdots \times n_3$. **Product Rule.**
Second rule: when order doesn’t matter divide..when possible.

Orderings of ANAGRAM?
Ordered Set: 7! First rule.
A’s are the same!
What is $\Delta$?
**ANAGRAM**
$A_1NA_2GRA_3M, A_2NA_1GRA_3M, \ldots$
Example: visualize

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide..when possible.

Orderings of ANAGRAM?
Ordered Set: 7! First rule.
A’s are the same!
What is \( \Delta \)?

\( \text{ANAGRAM} \)
\( A_1NA_2GRA_3M, A_2NA_1GRA_3M, \ldots \)
Example: visualize

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide..when possible.

Orderings of ANAGRAM?
Ordered Set: 7! First rule.
A’s are the same!
What is \( \Delta \)?
ANAGRAM
\( A_1NA_2GRA_3M, A_2NA_1GRA_3M, \ldots \)
\( \Delta = 3 \times 2 \times 1 \)
Example: visualize

First rule: \( n_1 \times n_2 \cdots \times n_3 \). **Product Rule.**
Second rule: when order doesn’t matter divide..when possible.

Orderings of ANAGRAM?
Ordered Set: 7! First rule.
A’s are the same!
What is \( \Delta \)?
ANAGRAM
\( A_1NA_2GRA_3M \), \( A_2NA_1GRA_3M \), ...
\( \Delta = 3 \times 2 \times 1 = 3! \)
First rule: \( n_1 \times n_2 \cdots \times n_3 \). **Product Rule.**
Second rule: when order doesn’t matter divide..when possible.

Orderings of **ANAGRAM**?
Ordered Set: 7! First rule.
A’s are the same!
What is \( \Delta \)?
**ANAGRAM**
\( A_1 \text{NA}_2 \text{GRA}_3 \text{M} , A_2 \text{NA}_1 \text{GRA}_3 \text{M} , ... \)
\( \Delta = 3 \times 2 \times 1 = 3! \) First rule!
Example: visualize

**First rule:** $n_1 \times n_2 \cdots \times n_3$. **Product Rule.**
**Second rule:** when order doesn’t matter divide...when possible.

Orderings of ANAGRAM?
Ordered Set: $7!$ First rule.
A’s are the same!
What is $\Delta$?

ANAGRAM
$A_1NA_2GRA_3M , A_2NA_1GRA_3M , ...$
$\Delta = 3 \times 2 \times 1 = 3!$ First rule!

$\Rightarrow \frac{7!}{3!}$
First rule: $n_1 \times n_2 \cdots \times n_3$. Product Rule.
Second rule: when order doesn’t matter divide..when possible.

Orderings of ANAGRAM?
Ordered Set: 7! First rule.
A’s are the same!
What is $\Delta$?

ANAGRAM
$A_1NA_2GRA_3M, A_2NA_1GRA_3M, \ldots$
$\Delta = 3 \times 2 \times 1 = 3!$ First rule!
$\Rightarrow \frac{7!}{3!}$ Second rule!
Summary.

$k$ Samples with replacement from $n$ items: $n^k$. 
$k$ Samples with replacement from $n$ items: $n^k$.
Sample without replacement: $\binom{n}{k} = \frac{n!}{(n-k)!}$.
$k$ Samples with replacement from $n$ items: $n^k$.
Sample without replacement: $\frac{n!}{(n-k)!}$.
Summary.

\[ k \text{ Samples with replacement from } n \text{ items: } n^k. \]
Sample without replacement: \[ \frac{n!}{(n-k)!} \]
Summary.

$k$ Samples with replacement from $n$ items: $n^k$.
Sample without replacement: $\frac{n!}{(n-k)!}$

Sample without replacement and order doesn’t matter: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.
“$n$ choose $k$”
Summary.

$k$ Samples with replacement from $n$ items: $n^k$.
Sample without replacement: $\frac{n!}{(n-k)!}$

Sample without replacement and order doesn’t matter: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.
“$n$ choose $k$”
(Count using first rule and second rule.)
Summary.

$k$ Samples with replacement from $n$ items: $n^k$.
Sample without replacement: $\frac{n!}{(n-k)!}$

Sample without replacement and order doesn’t matter: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.
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Sample without replacement and order doesn’t matter: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.
“$n$ choose $k$”
(Count using first rule and second rule.)

Sample with replacement and order doesn’t matter: $\binom{k+n-1}{n-1}$. 
Summary.

$k$ Samples with replacement from $n$ items: $n^k$.
Sample without replacement: $\frac{n!}{(n-k)!}$

Sample without replacement and order doesn’t matter: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.
“$n$ choose $k$”
(Count using first rule and second rule.)

Sample with replacement and order doesn’t matter: $\binom{k+n-1}{k+n-1}$.
Count with stars and bars:
Summary.

$k$ Samples with replacement from $n$ items: $n^k$.
Sample without replacement: $\frac{n!}{(n-k)!}$

Sample without replacement and order doesn’t matter: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.
“$n$ choose $k$”
(Count using first rule and second rule.)

Sample with replacement and order doesn’t matter: $\binom{k+n-1}{n-1}$.
Count with stars and bars:
how many ways to add up $n$ numbers to get $k$. 
Summary.

$k$ Samples with replacement from $n$ items: $n^k$.
Sample without replacement: $\frac{n!}{(n-k)!}$

Sample without replacement and order doesn’t matter: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.
“$n$ choose $k$”
(Count using first rule and second rule.)

Sample with replacement and order doesn’t matter: $\binom{k+n-1}{n-1}$.

Count with stars and bars:
  how many ways to add up $n$ numbers to get $k$.
  Each number is number of samples of type $i$
Summary.

$k$ Samples with replacement from $n$ items: $n^k$.
Sample without replacement: $\frac{n!}{(n-k)!}$

Sample without replacement and order doesn’t matter: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.
“$n$ choose $k$”
(Count using first rule and second rule.)

Sample with replacement and order doesn’t matter: $\binom{k+n-1}{n-1}$.

Count with stars and bars:
how many ways to add up $n$ numbers to get $k$.
Each number is number of samples of type $i$ which adds to total, $k$. 
Simple Inclusion/Exclusion

**Sum Rule:** For disjoint sets $S$ and $T$, $|S \cup T| = |S| + |T|$
Simple Inclusion/Exclusion

Sum Rule: For disjoint sets $S$ and $T$, $|S \cup T| = |S| + |T|$.

Example: How many permutations of $n$ items start with 1 or 2?
Simple Inclusion/Exclusion

Sum Rule: For disjoint sets \( S \) and \( T \), \(|S \cup T| = |S| + |T|\)

Example: How many permutations of \( n \) items start with 1 or 2? \( 1 \times (n - 1)! \)
Simple Inclusion/Exclusion

**Sum Rule:** For disjoint sets $S$ and $T$, $|S \cup T| = |S| + |T|$

**Example:** How many permutations of $n$ items start with 1 or 2?
$1 \times (n - 1)! + 1 \times (n - 1)!$
Simple Inclusion/Exclusion

Sum Rule: For disjoint sets $S$ and $T$, $|S \cup T| = |S| + |T|$

Example: How many permutations of $n$ items start with 1 or 2? $1 \times (n-1)! + 1 \times (n-1)!$

Inclusion/Exclusion Rule: For any $S$ and $T$, $|S \cup T| = |S| + |T| - |S \cap T|$. 
Simple Inclusion/Exclusion

**Sum Rule:** For disjoint sets $S$ and $T$, $|S \cup T| = |S| + |T|$

**Example:** How many permutations of $n$ items start with 1 or 2?
$1 \times (n - 1)! + 1 \times (n - 1)!$

**Inclusion/Exclusion Rule:** For any $S$ and $T$, $|S \cup T| = |S| + |T| - |S \cap T|.$

**Example:** How many 10-digit phone numbers have 7 as their first or second digit?
Simple Inclusion/Exclusion

**Sum Rule:** For disjoint sets $S$ and $T$, $|S \cup T| = |S| + |T|$

**Example:** How many permutations of $n$ items start with 1 or 2? $1 \times (n - 1)! + 1 \times (n - 1)!$

**Inclusion/Exclusion Rule:** For any $S$ and $T$, $|S \cup T| = |S| + |T| - |S \cap T|$.

**Example:** How many 10-digit phone numbers have 7 as their first or second digit?

$S =$ phone numbers with 7 as first digit.
Simple Inclusion/Exclusion

**Sum Rule:** For disjoint sets $S$ and $T$, $|S \cup T| = |S| + |T|$

**Example:** How many permutations of $n$ items start with 1 or 2? $1 \times (n - 1)! + 1 \times (n - 1)!$

**Inclusion/Exclusion Rule:** For any $S$ and $T$, $|S \cup T| = |S| + |T| - |S \cap T|$.

**Example:** How many 10-digit phone numbers have 7 as their first or second digit?

$S = \text{phone numbers with 7 as first digit.} |S| = 10^9$
Simple Inclusion/Exclusion

**Sum Rule:** For disjoint sets \( S \) and \( T \), \( |S \cup T| = |S| + |T| \)

**Example:** How many permutations of \( n \) items start with 1 or 2?
\[ 1 \times (n - 1)! + 1 \times (n - 1)! \]

**Inclusion/Exclusion Rule:** For any \( S \) and \( T \),
\[ |S \cup T| = |S| + |T| - |S \cap T| \]

**Example:** How many 10-digit phone numbers have 7 as their first or second digit?

\( S = \) phone numbers with 7 as first digit. \( |S| = 10^9 \)
\( T = \) phone numbers with 7 as second digit.
Simple Inclusion/Exclusion

**Sum Rule:** For disjoint sets $S$ and $T$, $|S \cup T| = |S| + |T|$

**Example:** How many permutations of $n$ items start with 1 or 2? 
$1 \times (n-1)! + 1 \times (n-1)!$

**Inclusion/Exclusion Rule:** For any $S$ and $T$, 
$|S \cup T| = |S| + |T| - |S \cap T|$.

**Example:** How many 10-digit phone numbers have 7 as their first or second digit? 
$S =$ phone numbers with 7 as first digit. $|S| = 10^9$
$T =$ phone numbers with 7 as second digit. $|T| = 10^9$. 
Sum Rule: For disjoint sets $S$ and $T$, $|S \cup T| = |S| + |T|$

Example: How many permutations of $n$ items start with 1 or 2?
$1 \times (n - 1)! + 1 \times (n - 1)!$

Inclusion/Exclusion Rule: For any $S$ and $T$, $|S \cup T| = |S| + |T| - |S \cap T|$.

Example: How many 10-digit phone numbers have 7 as their first or second digit?

$S =$ phone numbers with 7 as first digit. $|S| = 10^9$

$T =$ phone numbers with 7 as second digit. $|T| = 10^9$.

$S \cap T =$ phone numbers with 7 as first and second digit.
Simple Inclusion/Exclusion

Sum Rule: For disjoint sets $S$ and $T$, $|S \cup T| = |S| + |T|$

Example: How many permutations of $n$ items start with 1 or 2? $1 \times (n-1)! + 1 \times (n-1)!$

Inclusion/Exclusion Rule: For any $S$ and $T$, $|S \cup T| = |S| + |T| - |S \cap T|$. 

Example: How many 10-digit phone numbers have 7 as their first or second digit?

$S = $ phone numbers with 7 as first digit. $|S| = 10^9$

$T = $ phone numbers with 7 as second digit. $|T| = 10^9$.

$S \cap T = $ phone numbers with 7 as first and second digit. $|S \cap T| = 10^8$. 
Simple Inclusion/Exclusion

Sum Rule: For disjoint sets $S$ and $T$, $|S \cup T| = |S| + |T|$

Example: How many permutations of $n$ items start with 1 or 2?
$1 \times (n - 1)! + 1 \times (n - 1)!$

Inclusion/Exclusion Rule: For any $S$ and $T$, $|S \cup T| = |S| + |T| - |S \cap T|$.

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Answer: $|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$. 
Theorem: \( \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \).
Proof: How many size \( k \) subsets of \( n+1 \)?
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How many size \( k \) subsets of \( n+1 \)?
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So, \( \binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k} \).
Countability

Isomorphism principle.
Countability

Isomorphism principle.
Example.
Isomorphism principle.
Example.
Countability.
Isomorphism principle.
Example.
Countability.
Diagonalization.
Isomorphism principle.

Given a function, $f : D \rightarrow R$. 

$|D| = |R|$. 
Isomorphism principle.

Given a function, $f : D \rightarrow R$.

**One to One:**
Isomorphism principle.

Given a function, $f : D \rightarrow R$.

**One to One:**
For all $\forall x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.

**Onto:**
For all $y \in R$, $\exists x \in D$, $y = f(x)$.

$f(\cdot)$ is a bijection if it is one to one and onto.

Isomorphism principle:
If there is a bijection $f : D \rightarrow R$ then $|D| = |R|$. 
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Cardinalities of uncountable sets?

Cardinality of $[0, 1]$ smaller than all the reals?
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\(f : \mathbb{R}^+ \rightarrow [0, 1].\)
Cardinalities of uncountable sets?

Cardinality of $[0, 1]$ smaller than all the reals?

$f : \mathbb{R}^+ \rightarrow [0, 1]$.

$$f(x) = \begin{cases} 
  x + \frac{1}{2} & 0 \leq x \leq \frac{1}{2} \\
  \frac{1}{4x} & x > \frac{1}{2} 
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Bijection!

[0, 1] is same cardinality as nonnegative reals!
Countable.

Definition: $S$ is countable if there is a bijection between $S$ and some subset of $\mathbb{N}$. If the subset of $\mathbb{N}$ is finite, $S$ has finite cardinality. If the subset of $\mathbb{N}$ is infinite, $S$ is countably infinite. A bijection to or from the natural numbers implies countably infinite. Enumerable means countable. A subset of a countable set is countable. All countably infinite sets are the same cardinality as each other.
Countable.

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All countably infinite sets are the same cardinality as each other.
Examples

Countably infinite (same cardinality as naturals)

- $\mathbb{Z}^+$ - positive integers
Examples

Countably infinite (same cardinality as naturals)

- $\mathbb{Z}^+$ - positive integers
  Where’s 0?
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Countably infinite (same cardinality as naturals)

- $\mathbb{Z}^+$ - positive integers
  Where's 0?
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- $\mathbb{Z}$ - all integers.
  Twice as big?
  Bijection: $f(z) = 2|z| - \text{sign}(z)$.
Examples

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- \(E\) even numbers.
Examples

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Examples

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  Enumerate: 0,
Examples

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  Bijection: $f(z) = z - 1$.
  (Where’s 0? 1 Where’s 1? 2 ...)

- $E$ even numbers.
  Where are the odds? Half as big?
  Bijection: $f(e) = e/2$.

- $\mathbb{Z} - $ all integers.
  Twice as big?
  Bijection: $f(z) = 2|z| - \text{sign}(z)$.
  Enumerate: 0, −1,
Examples

Countably infinite (same cardinality as naturals)

- \( \mathbb{Z}^+ \) - positive integers
  Where’s 0?
  Bijection: \( f(z) = z - 1 \).
  (Where’s 0? 1 Where’s 1? 2 ...)

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Examples

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  Enumerate: 0, $-1, 1, -2, 2...$
Examples: Countable by enumeration

- $N \times N$ - Pairs of integers.
Examples: Countable by enumeration

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Examples: Countable by enumeration

- \( N \times N \) - Pairs of integers.
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  Enumerate: \((0, 0), (0, 1), (0, 2), \ldots \)
Examples: Countable by enumeration

- $\mathbb{N} \times \mathbb{N}$ - Pairs of integers.
  Square of countably infinite?
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  Never get to $(1,1)!$
  Enumerate: $(0,0)$,
Examples: Countable by enumeration

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  $(a, b)$ at position $(a + b - 1)(a + b)/2 + b$ in this order.
Examples: Countable by enumeration

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Examples: Countable by enumeration

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  Infinite Subset of pairs of natural numbers.
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- All rational numbers.
Examples: Countable by enumeration

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- Positive Rational numbers.
  Infinite Subset of pairs of natural numbers.
  Countably infinite.

- All rational numbers.
  Enumerate: list 0, positive and negative.
Examples: Countable by enumeration

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  Square of countably infinite?
  Enumerate: \((0, 0), (0, 1), (0, 2), \ldots ???
  Never get to \((1, 1)\)!
  Enumerate: \((0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2) \ldots
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- All rational numbers.
  Enumerate: list 0, positive and negative. How?
  Enumerate: 0, first positive, first negative, second positive..
  Will eventually get to any rational.
Diagonalization: power set of Integers.

The set of all subsets of $N$. 

\[ \text{Assume is countable.} \]
\[ \text{There is a listing, } L, \text{ that contains all subsets of } N. \]
\[ \text{Define a diagonal set, } D: \]
\[ \text{If } i \text{th set in } L \text{ does not contain } i, i \in D. \]
\[ \text{otherwise } i \notin D. \]
\[ D \text{ is different from } i \text{th set in } L \text{ for every } i. \]
\[ \Rightarrow D \text{ is not in the listing.} \]
\[ D \text{ is a subset of } N. \]
\[ L \text{ does not contain all subsets of } N. \]
\[ \text{Contradiction.} \]

Theorem: The set of all subsets of $N$ is not countable.

(The set of all subsets of $S$, is the powerset of $N$. )
Diagonalization: power set of Integers.

The set of all subsets of $N$.
Assume is countable.
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The set of all subsets of \( N \).
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There is a listing, \( L \), that contains all subsets of \( N \).
Diagonalization: power set of Integers.

The set of all subsets of \( N \).
Assume is countable.
There is a listing, \( L \), that contains all subsets of \( N \).
Define a diagonal set, \( D \):

\[ D = \{ i \in L : i \neq i \} \]

\( D \) is different from the \( i \)th set in \( L \) for every \( i \).
\( D \) is not in the listing.
\( D \) is a subset of \( N \).
\( L \) does not contain all subsets of \( N \).
Contradiction.

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The set of all subsets of $N$.

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Define a diagonal set, $D$:

If $i$th set in $L$ does not contain $i$, $i \in D$. 

$D$ is different from $i$th set in $L$ for every $i$.

$D$ is not in the listing.

$D$ is a subset of $N$.

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Halting problem is undecidable.
Uncomputability.

Halting problem is undecidable.
Diagonalization.
Uncomputability.

Halting problem is undecidable.
Diagonalization.
Halt does not exist.
Halt does not exist.

\[
\text{HALT}(P, I)
\]
Halt does not exist.

$HALT(P, I)$

$P$ - program
Halt does not exist.

$$HALT(P,I)$$
- $P$ - program
- $I$ - input.
Halt does not exist.

\[ \text{HALT}(P, I) \]

\[ P \] - program

\[ I \] - input.

Determines if \( P(I) \) (\( P \) run on \( I \)) halts or loops forever.
Halt does not exist.

\[ HALT(P, I) \]
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Determines if \( P(I) \) (\( P \) run on \( I \)) halts or loops forever.

**Theorem:** There is no program HALT.
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\[ P \text{ - program} \]
\[ I \text{ - input.} \]

Determines if \( P(I) \) (\( P \) run on \( I \)) halts or loops forever.

**Theorem:** There is no program HALT.

**Proof:** Yes!
Halt does not exist.

\[ \text{HALT}(P, I) \]
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Determines if \(P(I)\) (\(P\) run on \(I\)) halts or loops forever.

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$HALT(P, I)$

- $P$ - program
- $I$ - input.

Determines if $P(I)$ ($P$ run on $I$) halts or loops forever.

**Theorem:** There is no program HALT.

**Proof:** Yes! No! Yes! No!
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\[
HALT(P, I)
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P \text{ - program}
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Determines if \( P(I) \) (\( P \) run on \( I \)) halts or loops forever.

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- $P$ - program
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**Theorem:** There is no program $HALT$.

**Proof:** Yes! No! Yes! No! No! Yes!
Halt does not exist.

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**Proof:** Yes! No! Yes! No! No! Yes! No!
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\( \text{HALT}(P, I) \)

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Determines if \( P(I) \) (\( P \) run on \( I \)) halts or loops forever.

**Theorem:** There is no program HALT.

**Proof:** Yes! No! Yes! No! No! Yes! No! Yes! ..
Proof:

Assume there is a program $\text{HALT}(\cdot, \cdot)$. Turing(P)
1. If $\text{HALT}(P, P) = \text{halts}$, then go into an infinite loop.
2. Otherwise, halt immediately.

Assumption: there is a program $\text{HALT}$.

There is text that "is" the program $\text{HALT}$.

There is text that is the program $\text{Turing}$.

Can run $\text{Turing}$ on $\text{Turing}$!

Does $\text{Turing}(\text{Turing})$ halt?

$\text{Turing}(\text{Turing})$ halts
$\implies$ $\text{HALTS}(\text{Turing}, \text{Turing}) = \text{halts} = \implies$ $\text{Turing}(\text{Turing})$ loops forever.

$\text{Turing}(\text{Turing})$ loops forever
$\implies$ $\text{HALTS}(\text{Turing}, \text{Turing}) \neq \text{halts} = \implies$ $\text{Turing}(\text{Turing})$ halts.

Either way is contradiction. Program $\text{HALT}$ does not exist!
Halt and Turing.

**Proof:** Assume there is a program \( HALT(\cdot,\cdot) \).
Halt and Turing.

**Proof:** Assume there is a program $HALT(\cdot,\cdot)$.

Turing(P)
Halt and Turing.

**Proof:** Assume there is a program $HALT(·,·)$.

**Turing(P)**

1. If $HALT(P,P) =$"halts", then go into an infinite loop.
Halt and Turing.

**Proof:** Assume there is a program $HALT(\cdot,\cdot)$.

Turing($P$)
1. If $HALT(P,P) =$"halts", then go into an infinite loop.
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Halt and Turing.

**Proof:** Assume there is a program $HALT(\cdot,\cdot)$.

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**Proof:** Assume there is a program $HALT(\cdot, \cdot)$.

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1. If $HALT(P, P) =$ ”halts”, then go into an infinite loop.
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There is text that is the program Turing.
Halt and Turing.

Proof: Assume there is a program $HALT(P,P)$.

Turing($P$)
1. If $HALT(P,P) = \text{"halts"}$, then go into an infinite loop.
2. Otherwise, halt immediately.

Assumption: there is a program $HALT$.
There is text that “is” the program $HALT$.
There is text that is the program $Turing$.
Can run Turing on Turing!
Halt and Turing.

Proof: Assume there is a program $HALT(\cdot,\cdot)$.

Turing($P$)
1. If $HALT(P,P)$ = "halts", then go into an infinite loop.
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$\text{Turing(Turing)}$ halts
Halt and Turing.

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There is text that “is” the program HALT.
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Can run Turing on Turing!

Does $Turing(Turing)$ halt?

$Turing(Turing)$ halts
$\implies$ then $HALTS(Turing, Turing) = halts$
Halt and Turing.

Proof: Assume there is a program $HALT(\cdot, \cdot)$.

$Turing(P)$
1. If $HALT(P,P) =$"halts", then go into an infinite loop.
2. Otherwise, halt immediately.

Assumption: there is a program $HALT$.
There is text that “is” the program $HALT$.
There is text that is the program $Turing$.
Can run $Turing$ on $Turing$!

Does $Turing(Turing)$ halt?

$Turing(Turing)$ halts
$\implies$ then $HALTS(Turing, Turing) = halts$
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**Halt and Turing.**

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Either way is contradiction. Program \( \text{HALT} \) does not exist!
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Any program is a fixed length string.
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Halt does not exist!
Undecidable problems.

Does a program print “Hello World”? 

Find exit points and add statement: Print “Hello World.”

Can a set of notched tiles tile the infinite plane? Proof: simulate a computer. Halts if finite.

Does a set of integer equations have a solution? Example: Ask program if $x^n + y^n = 1$? has integer solutions. Problem is undecidable.

Be careful!

Is there a solution to $x^n + y^n = 1$? (Diophantine equation.) The answer is yes or no. This “problem” is not undecidable.

Undecidability for Diophantine set of equations $\Rightarrow$ no program can take any set of integer equations and always output correct answer.
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Midterm format

Time: approximately 120 minutes.
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Many short answers.
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Know material medium: slower, less correct.

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Some longer questions.

Priming: sequence of questions...

but don't overdo this as test strategy!!!

Ideas, conceptual, more calculation.
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Priming: sequence of questions...
  but don’t overdo this as test strategy!!

Ideas, conceptual,
  more calculation.
Midterm format

Time: approximately 120 minutes.

Many short answers.
   Get at ideas that we study.
      Know material well: fast, correct.
      Know material medium: slower, less correct.
      Know material not so well: Uh oh.

Some longer questions.

Priming: sequence of questions...
   but don’t overdo this as test strategy!!!
Wrapup.
Wrapup.

Watch Piazza for Logistics!

satishr@cs.berkeley.edu, admin@cs70.org

Good Studying!
Wrapup.

Watch Piazza for Logistics!
Watch Piazza for Advice!
Wrapup.

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If you sent me email about Midterm conflicts
Wrapup.

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If you sent me email about Midterm conflicts
Other arrangements.
Wrapup.

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Other issues....
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