CS70: Lecture 22.

Part I: Confidence Intervals Again

Part II: Linear Regression
1. Confidence?

2. Example

3. Review of Chebyshev

4. Confidence Interval with Chebyshev

5. More examples
Confidence?

You flip a coin once and get H. Do you think that \( \Pr[H] = 1 \)?

You flip a coin 10 times and get 5 Hs. Are you sure that \( \Pr[H] = 0.5 \)?

You flip a coin 106 times and get 35% of Hs. How much are you willing to bet that \( \Pr[H] \) is exactly 0.35? How much are you willing to bet that \( \Pr[H] \in [0.3, 0.4] \)?

Did different exam rooms perform differently? (6 afraid of 7?)

More generally, you estimate an unknown quantity \( \theta \). Your estimate is \( \hat{\theta} \). How much confidence do you have in your estimate?
Confidence?

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Confidence is essential in many applications:
▶ How effective is a medication?
▶ Are we sure of the mileage of a car?
▶ Can we guarantee the lifespan of a device?
▶ We simulated a system. Do we trust the simulation results?
▶ Is an algorithm guaranteed to be fast?
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As scientists and engineers, be convinced of this fact:
An estimate without confidence level is useless!
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An estimate without confidence level is useless!
The following definition captures precisely the notion of confidence.

Definition: Confidence Interval

An interval \([a, b]\) is a 95\% confidence interval for an unknown quantity \(\theta\) if \(\Pr[\theta \in [a, b]] \geq 95\%\).

The interval \([a, b]\) is calculated on the basis of observations. Here is a typical framework.

Assume that \(X_1, X_2, \ldots, X_n\) are i.i.d. and have a distribution that depends on some parameter \(\theta\).

For instance, \(X_n = B(\theta)\).

Thus, more precisely, given \(\theta\), the random variables \(X_n\) are i.i.d. with a known distribution (that depends on \(\theta\)).

\[\text{We observe} X_1, \ldots, X_n\]
\[\text{We calculate} a = a(X_1, \ldots, X_n)\text{ and } b = b(X_1, \ldots, X_n)\]

\[\text{If we can guarantee that} \Pr[\theta \in [a, b]] \geq 95\%, \text{then} [a, b] \text{ is a 95\% CI for} \theta.\]
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- If we can guarantee that \(Pr[\theta \in [a, b]] \geq 95\%\), then \([a, b]\) is a 95\%-CI for \(\theta\).
Confidence Interval: Applications

We poll 1000 people.

Among those, 48% declare they will vote for Trump.

We do some calculations ... 

We conclude that $[0.43, 0.53]$ is a 95% CI for the fraction of all the voters who will vote for Trump.

We observe 1,000 heart valve replacements that were performed by Dr. Bill.

Among those, 35 patients died during surgery. (Sad example!)

We do some calculations ...

We conclude that $[1\%, 5\%]$ is a 95% CI for the probability of dying during that surgery by Dr. Bill.

We do a similar calculation for Dr. Fred.

We find that $[8\%, 12\%]$ is a 95% CI for Dr. Fred's surgery.

What surgeon do you choose?
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  - What surgeon do you choose?
Coin Flips: Intuition

Say that you flip a coin \( n = 100 \) times and observe 20 Hs. If \( p := \Pr[H] = 0.5 \), this event is very unlikely. Intuitively, if is unlikely that the fraction of Hs, say \( A_n \), differs a lot from \( p := \Pr[H] \). Thus, it is unlikely that \( p \) differs a lot from \( A_n \). Hence, one should be able to build a confidence interval \([A_n - \varepsilon, A_n + \varepsilon]\) for \( p \).

The key idea is that \(|A_n - p| \leq \varepsilon \iff p \in [A_n - \varepsilon, A_n + \varepsilon]\). Thus, \( \Pr[|A_n - p| > \varepsilon] \leq 5\% \iff \Pr[p \in [A_n - \varepsilon, A_n + \varepsilon]] \geq 95\% \).

It remains to find \( \varepsilon \) such that \( \Pr[|A_n - p| > \varepsilon] \leq 5\% \).

One approach: Chebyshev.
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The key idea is that \( |\frac{A_n}{n} - p| \leq \varepsilon \iff p \in \left[\frac{A_n}{n} - \varepsilon, \frac{A_n}{n} + \varepsilon\right] \).

Thus, \( \Pr[|\frac{A_n}{n} - p| > \varepsilon] \leq 5\% \iff \Pr[p \in \left[\frac{A_n}{n} - \varepsilon, \frac{A_n}{n} + \varepsilon\right]] \geq 95\% \).

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![Histogram of binomial distributions for different probabilities]
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The key idea is that $|A_n - p| \leq \varepsilon \iff p \in [A_n - \varepsilon, A_n + \varepsilon]$.

Thus, $Pr[|A_n - p| > \varepsilon] \leq 5\% \iff Pr[p \in [A_n - \varepsilon, A_n + \varepsilon]] \geq 95\%$. 
Coin Flips: Intuition

Say that you flip a coin \( n = 100 \) times and observe 20 Hs.

If \( p := Pr[H] = 0.5 \), this event is very unlikely.

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Thus, it is unlikely that \( p \) differs a lot from \( A_n \). Hence, one should be able to build a confidence interval \( [A_n - \varepsilon, A_n + \varepsilon] \) for \( p \).

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One approach: Chebyshev.
Confidence Interval with Chebyshev

Flip a coin \( n \) times. Let \( A_n \) be the fraction of \( H \)s.

Can we find \( \epsilon \) such that
\[
\Pr \left[ |A_n - p| > \epsilon \right] \leq 5\%.
\]

Using Chebyshev, we will see that \( \epsilon = \frac{2}{\sqrt{n}} \) works.

Thus
\[
\left[ A_n - \frac{2}{\sqrt{n}}, A_n + \frac{2}{\sqrt{n}} \right]
\]
is a 95%-CI for \( p \).

Example: If \( n = 1500 \), then
\[
\Pr \left[ p \in \left[ A_n - 0.05, A_n + 0.05 \right] \right] \geq 95\%.
\]

In fact, \( \alpha = \frac{1}{\sqrt{n}} \) works, so that with \( n = 1500 \), one has
\[
\Pr \left[ p \in \left[ A_n - 0.02, A_n + 0.02 \right] \right] \geq 95\%.
\]
Confidence Interval with Chebyshev

- Flip a coin $n$ times.
Confidence Interval with Chebyshev

- Flip a coin \( n \) times. Let \( A_n \) be the fraction of \( Hs \).
Confidence Interval with Chebyshev

- Flip a coin $n$ times. Let $A_n$ be the fraction of $H$s.
- Can we find $\varepsilon$ such that $Pr[|A_n - \rho| > \varepsilon] \leq 5\%$?
Confidence Interval with Chebyshev

- Flip a coin \( n \) times. Let \( A_n \) be the fraction of \( Hs \).
- Can we find \( \varepsilon \) such that \( Pr[|A_n - p| > \varepsilon] \leq 5\% \)?

Using Chebyshev, we will see that \( \varepsilon = 2.25 \frac{1}{\sqrt{n}} \) works.
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Example: If $n = 1500$, then $Pr[p \in [A_n - 0.05, A_n + 0.05]] \geq 95\%$. 
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In fact, \( a = \frac{1}{\sqrt{n}} \) works, so that with \( n = 1,500 \) one has

\( Pr[p \in [A_n - 0.02, A_n + 0.02]] \geq 95\% \).
Theorem:

Let \( X_n \) be i.i.d. with mean \( \mu \) and variance \( \sigma^2 \).

Define \( A_n = X_1 + \cdots + X_n \).

Then, \( \Pr\left[ \mu \in \left[ A_n - 4 \cdot \frac{\sigma}{\sqrt{n}}, A_n + 4 \cdot \frac{\sigma}{\sqrt{n}} \right] \right] \geq 95\% \).

Thus, \( \left[ A_n - 4 \cdot \frac{\sigma}{\sqrt{n}}, A_n + 4 \cdot \frac{\sigma}{\sqrt{n}} \right] \) is a 95\% CI for \( \mu \).

Example:

Let \( X_n = 1 \{ \text{coin yields H} \} \).

Then \( \mu = \mathbb{E}[X_n] = p : = \Pr[H] \).

Also, \( \sigma^2 = \text{var}(X_n) = p(1-p) \leq \frac{1}{4} \).

Hence, \( \left[ A_n - 4 \cdot \frac{1}{\sqrt{2n}}, A_n + 4 \cdot \frac{1}{\sqrt{2n}} \right] \) is a 95\% CI for \( p \).
Theorem:
Let $X_n$ be i.i.d. with mean $\mu$ and variance $\sigma^2$. 
Confidence Intervals: Result

**Theorem:**
Let $X_n$ be i.i.d. with mean $\mu$ and variance $\sigma^2$.
Define $A_n = \frac{X_1 + \cdots + X_n}{n}$. 

Thus, $\left[ A_n - 4.5\sigma\sqrt{\frac{1}{n}}, A_n + 4.5\sigma\sqrt{\frac{1}{n}} \right]$ is a 95% CI for $\mu$.

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Let $X_n = 1\{\text{coin } n \text{ yields } H\}$.
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Thus, $[A_n - 4.5 \frac{\sigma}{\sqrt{n}}, A_n + 4.5 \frac{\sigma}{\sqrt{n}}]$ is a 95\%-CI for $\mu$.

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Also, $\sigma^2 = \text{var}(X_n) = p(1 - p) \leq \frac{1}{4}$.

Hence, $[A_n - 4.5 \frac{1/2}{\sqrt{n}}, A_n + 4.5 \frac{1/2}{\sqrt{n}}]$ is a 95\%-CI for $p$. 
Confidence Interval: Analysis

We prove the theorem, i.e., that $A_n \pm 4.5 \frac{\sigma}{\sqrt{n}}$ is a 95\% CI for $\mu$.

From Chebyshev:

$$\Pr \left[ |A_n - \mu| \geq 4.5 \frac{\sigma}{\sqrt{n}} \right] \leq \frac{\text{var}(A_n)}{4.5^2 \frac{\sigma^2}{n}} = \frac{\sigma^2}{n} \cdot \frac{1}{4.5^2} \cdot \frac{n}{\sigma^2} = \frac{1}{20} = 5\%.$$ 

Thus,

$$\Pr \left[ |A_n - \mu| \leq 4.5 \frac{\sigma}{\sqrt{n}} \right] \geq 95\%.$$ 

Hence,

$$\Pr \left[ \mu \in \left[ A_n - 4.5 \frac{\sigma}{\sqrt{n}}, A_n + 4.5 \frac{\sigma}{\sqrt{n}} \right] \right] \geq 95\%.$$
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= \frac{1}{n^2} \times n \cdot \text{var}(X_1) = \frac{\sigma^2}{n}
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Confidence Interval: Analysis

We prove the theorem, i.e., that $A_n \pm 4.5\sigma / \sqrt{n}$ is a 95%-CI for $\mu$.

From Chebyshev:

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Hence,

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Thus,

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Hence,

$$Pr[\mu \in [A_n - 4.5\sigma / \sqrt{n}, A_n + 4.5\sigma / \sqrt{n}]] \geq 95\%.$$
Confidence interval for $p$ in $B(p)$
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Let $X_n$ be i.i.d. $B(p)$. 
Confidence interval for $p$ in $B(p)$

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Confidence interval for $p$ in $B(p)$

Let $X_n$ be i.i.d. $B(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

**Theorem:**

$$[A_n - \frac{2.25}{\sqrt{n}}, A_n + \frac{2.25}{\sqrt{n}}]$$ is a 95%-CI for $p$. 
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**Proof:**
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Proof:

We have just seen that

$$Pr[\mu \in [A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}]] \geq 95\%.$$
Confidence interval for \( p \) in \( B(p) \)

Let \( X_n \) be i.i.d. \( B(p) \). Define \( A_n = (X_1 + \cdots + X_n)/n \).

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We have just seen that

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Here, \( \mu = p \)
Confidence interval for \( p \) in \( B(p) \)

Let \( X_n \) be i.i.d. \( B(p) \). Define \( A_n = (X_1 + \cdots + X_n)/n \).

**Theorem:**

\[
[A_n - \frac{2.25}{\sqrt{n}}, A_n + \frac{2.25}{\sqrt{n}}]
\]

is a 95\%-CI for \( p \).

**Proof:**

We have just seen that

\[
Pr[\mu \in [A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}]] \geq 95%.
\]

Here, \( \mu = p \) and \( \sigma^2 = p(1-p) \).
Confidence interval for $p$ in $B(p)$

Let $X_n$ be i.i.d. $B(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

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Here, $\mu = p$ and $\sigma^2 = p(1-p)$. Thus, $\sigma^2 \leq \frac{1}{4}$
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Here, $\mu = p$ and $\sigma^2 = p(1-p)$. Thus, $\sigma^2 \leq \frac{1}{4}$ and $\sigma \leq \frac{1}{2}$.

Thus,

$$Pr[\mu \in [A_n - 4.5 \times 0.5/\sqrt{n}, A_n + 4.5 \times 0.5/\sqrt{n}]] \geq 95\%.$$
Confidence interval for $p$ in $B(p)$

Let $X_n$ be i.i.d. $B(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

**Theorem:**

$$[A_n - \frac{2.25}{\sqrt{n}}, A_n + \frac{2.25}{\sqrt{n}}]$$

is a 95%-CI for $p$.

**Proof:**

We have just seen that

$$Pr[\mu \in [A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}] \geq 95\%.$$ 

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Confidence interval for $p$ in $B(p)$
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An illustration:
Confidence interval for $p$ in $B(p)$

An illustration:

\[ 95\% - \text{CI for } p = \left[ A_n - 2.25 \frac{1}{\sqrt{n}}, A_n + 2.25 \frac{1}{\sqrt{n}} \right] \]
Confidence interval for $p$ in $B(p)$

An illustration:

Good practice: You run your simulation, or experiment.
Confidence interval for $p$ in $B(p)$

An illustration:

$95\% - \text{CI for } p$

$= [A_n - 2.25 \frac{1}{\sqrt{n}}, A_n + 2.25 \frac{1}{\sqrt{n}}]$

Good practice: You run your simulation, or experiment. You get an estimate.
Confidence interval for $p$ in $B(p)$

An illustration:

$95\% - CI$ for $p$

$= [A_n - 2.25 \frac{1}{\sqrt{n}}, A_n + 2.25 \frac{1}{\sqrt{n}}]$}

Good practice: You run your simulation, or experiment. You get an estimate. You indicate your confidence interval.
Confidence interval for $p$ in $B(p)$
Confidence interval for $p$ in $B(p)$

Improved CI:
Confidence interval for $p$ in $B(p)$

Improved CI: In fact, one can replace 2.25 by 1.
Confidence interval for $p$ in $B(p)$

Improved CI: In fact, one can replace $2.25$ by $1$.

Quite a bit of work to get there:
Confidence interval for $p$ in $B(p)$

Improved CI: In fact, one can replace 2.25 by 1.

Quite a bit of work to get there: continuous random variables;
Confidence interval for $p$ in $B(p)$

Improved CI: In fact, one can replace 2.25 by 1.

Quite a bit of work to get there: continuous random variables; Gaussian;
Confidence interval for $p$ in $B(p)$

Improved CI: In fact, one can replace 2.25 by 1.

Quite a bit of work to get there: continuous random variables; Gaussian; Central Limit Theorem.
Confidence Interval for $1/p$ in $G(p)$

Let $X_i$ be i.i.d. $G(p)$. Define $A_n = \frac{X_1 + \cdots + X_n}{n}$.

**Theorem:** $[A_n - 4 \cdot \frac{1}{p} \sqrt{\frac{1}{n}}, A_n + 4 \cdot \frac{1}{p} \sqrt{\frac{1}{n}}]$ is a 95%-CI for $1/p$.

**Proof:** We know that $\Pr[\mu \in [A_n - 4 \cdot \frac{1}{p} \sqrt{\frac{1}{n}}, A_n + 4 \cdot \frac{1}{p} \sqrt{\frac{1}{n}}]] \geq 95\%$. Here, $\mu = \frac{1}{p}$ and $\sigma = \sqrt{\frac{1}{p} \cdot \frac{1}{1-p}} \leq \frac{1}{p}$. Hence, $\Pr\left[\frac{1}{p} \in \left[A_n - 4 \cdot \frac{1}{p} \sqrt{\frac{1}{n}}, A_n + 4 \cdot \frac{1}{p} \sqrt{\frac{1}{n}}\right]\right] \geq 95\%$.

Now, $A_n - 4 \cdot \frac{1}{p} \sqrt{\frac{1}{n}} \leq \frac{1}{p} \leq A_n + 4 \cdot \frac{1}{p} \sqrt{\frac{1}{n}}$ is equivalent to $A_n + 4 \cdot \frac{1}{p} \sqrt{\frac{1}{n}} \leq \frac{1}{p} \leq A_n + 4 \cdot \frac{1}{p} \sqrt{\frac{1}{n}}$.

**Examples:** $[0.7, 1.0]_{100}$ and $[0.96, 1.05]_{10000}$. 
Confidence Interval for $1/p$ in $G(p)$

Let $X_n$ be i.i.d. $G(p)$. 

Confidence Interval for $1/p$ in $G(p)$

Let $X_n$ be i.i.d. $G(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$. 

**Theorem:**

\[
A_n - 4.5 \frac{1}{\sqrt{n}}, \quad A_n + 4.5 \frac{1}{\sqrt{n}}
\]

is a 95%-CI for $1/p$.

**Proof:**

We know that

\[
\Pr[A_n - 4.5 \frac{1}{\sqrt{n}} \leq \frac{1}{p} \leq A_n + 4.5 \frac{1}{\sqrt{n}}] \geq 95\%.
\]

Here, $\mu = 1/p$ and $\sigma = \sqrt{1/p - p}$ $\leq 1/p$.

Hence,

\[
\Pr\left[\frac{1}{p} \in \left(A_n - 4.5 \frac{1}{\sqrt{n}}, A_n + 4.5 \frac{1}{\sqrt{n}}\right)\right] \geq 95\%.
\]

Now,

\[
A_n - 4.5 \frac{1}{\sqrt{n}} \leq \frac{1}{p} \leq A_n + 4.5 \frac{1}{\sqrt{n}}
\]

is equivalent to

\[
A_n - 4.5 \frac{1}{\sqrt{n}} \leq 1/p \leq A_n + 4.5 \frac{1}{\sqrt{n}}.
\]

**Examples:**

\[
[0.7 A_{100}, 1.8 A_{100}], \quad [0.96 A_{10000}, 1.05 A_{10000}].
\]
Confidence Interval for $1/p$ in $G(p)$

Let $X_n$ be i.i.d. $G(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

Theorem:

$$\left[ \frac{A_n}{1 + 4.5/\sqrt{n}}, \frac{A_n}{1 - 4.5/\sqrt{n}} \right]$$

is a 95%-CI for $1/p$. 

Examples:

\[ \left[ 0.7 A_{100}, 1.8 A_{100} \right] \text{ and } \left[ 0.96 A_{10000}, 1.05 A_{10000} \right]. \]
Confidence Interval for $1/p$ in $G(p)$

Let $X_n$ be i.i.d. $G(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

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Confidence Interval for $1/p$ in $G(p)$

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is a 95%-CI for $\frac{1}{p}$.

**Proof:** We know that

$$Pr[\mu \in [A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}]] \geq 95\%.$$
Confidence Interval for $1/p$ in $G(p)$

Let $X_n$ be i.i.d. $G(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

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$$Pr[\mu \in [A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}]] \geq 95%.$$  

Here, $\mu = \frac{1}{p}$ and $\sigma = \frac{\sqrt{1-p}}{p} \leq \frac{1}{p}$. 
Confidence Interval for $1/p$ in $G(p)$

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Here, $\mu = \frac{1}{p}$ and $\sigma = \frac{\sqrt{1-p}}{p} \leq \frac{1}{p}$. Hence,

$$Pr[\frac{1}{p} \in [A_n - 4.5 \frac{1}{p\sqrt{n}}, A_n + 4.5 \frac{1}{p\sqrt{n}}] \geq 95\%.$$


Confidence Interval for $1/p$ in $G(p)$

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Here, $\mu = \frac{1}{p}$ and $\sigma = \frac{\sqrt{1-p}}{p} \leq \frac{1}{p}$. Hence,

$$Pr[\frac{1}{p} \in [A_n - 4.5 \frac{1}{p\sqrt{n}}, A_n + 4.5 \frac{1}{p\sqrt{n}}]] \geq 95\%.$$

Now, $A_n - 4.5 \frac{1}{p\sqrt{n}} \leq \frac{1}{p} \leq \frac{1}{p} \leq A_n + 4.5 \frac{1}{p\sqrt{n}}$ is equivalent to
Confidence Interval for $1/p$ in $G(p)$

Let $X_n$ be i.i.d. $G(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

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is a 95%-CI for $\frac{1}{p}$.

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$$Pr[\mu \in [A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}]] \geq 95\%.$$

Here, $\mu = \frac{1}{p}$ and $\sigma = \frac{\sqrt{1-p}}{p} \leq \frac{1}{p}$. Hence,

$$Pr\left[ \frac{1}{p} \in [A_n - 4.5 \frac{1}{p\sqrt{n}}, A_n + 4.5 \frac{1}{p\sqrt{n}}] \right] \geq 95\%.$$

Now, $A_n - 4.5 \frac{1}{p\sqrt{n}} \leq \frac{1}{p} \leq A_n + 4.5 \frac{1}{p\sqrt{n}}$ is equivalent to

$$\frac{A_n}{1 + 4.5/\sqrt{n}} \leq \frac{1}{p} \leq \frac{A_n}{1 - 4.5/\sqrt{n}}.$$
Confidence Interval for $1/p$ in $G(p)$

Let $X_n$ be i.i.d. $G(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

**Theorem:**

\[
\left[ \frac{A_n}{1 + 4.5/\sqrt{n}}, \frac{A_n}{1 - 4.5/\sqrt{n}} \right]
\]

is a 95%-CI for $\frac{1}{p}$.

**Proof:** We know that

\[
Pr[\mu \in [A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}]] \geq 95%.
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Here, $\mu = \frac{1}{p}$ and $\sigma = \frac{\sqrt{1-p}}{p} \leq \frac{1}{p}$. Hence,

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Pr[\frac{1}{p} \in [A_n - 4.5 \frac{1}{p\sqrt{n}}, A_n + 4.5 \frac{1}{p\sqrt{n}}]] \geq 95%.
\]

Now, $A_n - 4.5 \frac{1}{p\sqrt{n}} \leq \frac{1}{p} \leq A_n + 4.5 \frac{1}{p\sqrt{n}}$ is equivalent to

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**Examples:**

\[
\left[ 0.7 A_{100}, 1.8 A_{100} \right] \text{ and } \left[ 0.96 A_{10000}, 1.05 A_{10000} \right].
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**Theorem:** 

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**Proof:** We know that 

$$Pr[\mu \in [A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}]] \geq 95\%.$$  

Here, $\mu = \frac{1}{p}$ and $\sigma = \frac{\sqrt{1-p}}{p} \leq \frac{1}{p}$. Hence, 

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**Examples:** $[0.7A_{100}, 1.8A_{100}]$ and $[0.96A_{10000}, 1.05A_{10000}]$. 
Which Coin is Better?

You are given coin $A$ and coin $B$. You want to find out which one has a larger $\Pr[H]$. Let $p_A$ and $p_B$ be the values of $\Pr[H]$ for the two coins.

Approach:

▶ Flip each coin $n$ times.
▶ Let $A_n$ be the fraction of Hs for coin $A$ and $B_n$ for coin $B$.
▶ Assume $A_n > B_n$. It is tempting to think that $p_A > p_B$.

Confidence?

Analysis:

Note that $E[A_n - B_n] = p_A - p_B$ and $\text{var}(A_n - B_n) = \frac{1}{n}(p_A(1-p_A) + p_B(1-p_B)) \leq \frac{1}{2n}$.

Thus, $\Pr[|A_n - B_n - (p_A - p_B)| > \varepsilon] \leq \frac{1}{2n \varepsilon^2}$, so $\Pr[p_A - p_B \in [A_n - B_n - \varepsilon, A_n - B_n + \varepsilon]] \geq 1 - \frac{1}{2n \varepsilon^2}$.

Example:

With $n = 100$ and $A_n - B_n = 0.2$, $\Pr[p_A > p_B] \geq 1 - \frac{1}{8} = 0.875$. 
Which Coin is Better?

You are given coin A and coin B.

Let $p_A$ and $p_B$ be the values of $P[H]$ for the two coins.

**Approach:**

- Flip each coin $n$ times.
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**Analysis:**

Note that $E[A_n - B_n] = p_A - p_B$ and $\text{var}(A_n - B_n) = \frac{1}{n}(p_A(1-p_A) + p_B(1-p_B)) \leq \frac{1}{2n}$.

Thus, $\Pr[|A_n - B_n - (p_A - p_B)| > \epsilon] \leq \frac{1}{2} \epsilon^2$, so $\Pr[p_A - p_B \in [A_n - B_n - \epsilon, A_n - B_n + \epsilon]] \geq 1 - \frac{1}{2n} \epsilon^2$, and $\Pr[p_A - p_B \geq 0] \geq 1 - \frac{1}{2n} (A_n - B_n)^2$.

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You are given coin $A$ and coin $B$. You want to find out which one has a larger $Pr[H]$. 

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Note that $E[A_n - B_n] = p_A - p_B$ and $\text{var}(A_n - B_n) = \frac{1}{n}(p_A(1-p_A) + p_B(1-p_B)) \leq \frac{1}{2n}$.

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**Confidence?**

**Analysis:** Note that

$$E[A_n - B_n] =$$
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**Confidence?**

**Analysis:** Note that

$$E[A_n - B_n] = p_A - p_B$$

and

$$\text{var}(A_n - B_n) =$$
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**Analysis:** Note that

$$E[A_n - B_n] = p_A - p_B$$

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$$\text{var}(A_n - B_n) = \frac{1}{n}(p_A(1-p_A) + p_B(1-p_B)) \leq \frac{1}{2n}.$$
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You are given coin A and coin B. You want to find out which one has a larger Pr[H]. Let $p_A$ and $p_B$ be the values of Pr[H] for the two coins.

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- Let $A_n$ be the fraction of Hs for coin A and $B_n$ for coin B.
- Assume $A_n > B_n$. It is tempting to think that $p_A > p_B$.

**Confidence?**

**Analysis:** Note that

\[ E[A_n - B_n] = p_A - p_B \text{ and } var(A_n - B_n) = \frac{1}{n}(p_A(1 - p_A) + p_B(1 - p_B)) \leq \frac{1}{2n}. \]

Thus, $Pr[|A_n - B_n - (p_A - p_B)| > \varepsilon] \leq \frac{1}{2n\varepsilon^2}$,
Which Coin is Better?

You are given coin $A$ and coin $B$. You want to find out which one has a larger $Pr[H]$. Let $p_A$ and $p_B$ be the values of $Pr[H]$ for the two coins.

**Approach:**

- Flip each coin $n$ times.
- Let $A_n$ be the fraction of Hs for coin $A$ and $B_n$ for coin $B$.
- Assume $A_n > B_n$. It is tempting to think that $p_A > p_B$.

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Thus, $Pr[|A_n - B_n - (p_A - p_B)| > \varepsilon] \leq \frac{1}{2n\varepsilon^2}$, so

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**Example:**

With $n = 100$ and $A_n - B_n = 0.2$, $Pr[p_A > p_B] \geq 1 - \frac{1}{8} = 0.875$. 

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For $B(p)$, we wanted to estimate $p$. The CI requires $\sigma = \sqrt{p(1-p)}$. We replaced $\sigma$ by an upper bound: $1/2$. In some applications, it may be OK to replace $\sigma^2$ by the following sample variance: $s^2_n := \frac{1}{n} \sum_{m=1}^{n} (X_m - A_n)^2$. However, in some cases, this is dangerous! The theory says it is OK if the distribution of $X_n$ is nice (Gaussian). This is used regularly in practice. However, be aware of the risk.
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Confidence Intervals

1. Estimates without confidence level are useless!
2. \([a, b]\) is a 95\% CI for \(\theta\) if \(\Pr[\theta \in [a, b]] \geq 95\%\).
3. Using Chebyshev: \([A_n - 4\frac{\sigma}{\sqrt{n}}, A_n + 4\frac{\sigma}{\sqrt{n}}]\) is a 95\% CI for \(\mu\).
4. Using CLT, we will replace 4\frac{\sigma}{\sqrt{n}} by 2\frac{\sigma}{\sqrt{n}}.
5. When \(\sigma\) is not known, one can replace it by an upper bound.
6. Examples: \(B(p), G(p)\), which coin is better?
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Summary

**Confidence Intervals**

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2. \([a, b]\) is a 95\%-CI for \(\theta\) if \(Pr[\theta \in [a, b]] \geq 95\%\).
3. Using Chebyshev: \([A_n - 4.5\sigma / \sqrt{n}, A_n + 4.5\sigma / \sqrt{n}]\) is a 95\%-CI for \(\mu\).
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Linear Regression.

Linear Regression

1. Preamble
2. Motivation for LR
3. History of LR
4. Linear Regression
5. Derivation
6. More examples
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The best guess about $Y$, if we know only the distribution of $Y$, is $E[ Y ]$. More precisely, the value of $a$ that minimizes $E[ (Y - a)^2 ]$ is $a = E[ Y ]$.

Proof:
Let $\hat{Y} := Y - E[ Y ]$. Then, $E[ \hat{Y} ] = 0$. So, $E[ \hat{Y} c ] = 0$ for all $c$. Now,


with $c = E[ Y ] - a = E[ \hat{Y} ]$. Hence,

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Hence, $E[(Y - a)^2] \geq E[(Y - E[Y])^2], \forall a.$
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Hence, $E[(Y - a)^2] \geq E[(Y - E[Y])^2], \forall a.$

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Thus, if we want to guess the value of $Y$, we choose $E[Y]$. Now assume we make some observation $X$ related to $Y$. How do we use that observation to improve our guess about $Y$? The idea is to use a function $g(X)$ of the observation to estimate $Y$. The simplest function $g(X)$ is a constant that does not depend on $X$. The next simplest function is linear: $g(X) = a + bX$. What is the best linear function? That is our next topic. A bit later, we will consider a general function $g(X)$. 
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Linear Regression: Preamble
Linear Regression: Motivation

Example 1: 100 people.

Let $(X_n, Y_n) = (\text{height}, \text{weight})$ of person $n$, for $n = 1, \ldots, 100$:

The blue line is $Y = -114.3 + 106.5X$. ($X$ in meters, $Y$ in kg.)

Best linear fit: Linear Regression.
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![Fitted Line Plot](image)

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Example 2: 15 people.
Motivation

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We look at two attributes: \((X_n, Y_n)\) of person \(n\), for \(n = 1, \ldots, 15:\)
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The line \(Y = a + bX\) is the linear regression.
Covariance

**Definition** The covariance of $X$ and $Y$ is

$$cov(X, Y) := E[(X - E[X])(Y - E[Y])].$$
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**Fact**

$$cov(X, Y) = E[XY] - E[X]E[Y].$$

**Proof:**
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**Proof:**
Think about $E[X] = E[Y] = 0$. Just $E[XY]$. \( \square \)

For the sake of completeness.
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$$= E[XY] - E[X]E[Y].$$
Examples of Covariance

Note that $E[X] = 0$ and $E[Y] = 0$ in these examples. Then $\text{cov}(X, Y) = E[XY]$.

When $\text{cov}(X, Y) > 0$, the RVs $X$ and $Y$ tend to be large or small together. $X$ and $Y$ are said to be positively correlated.

When $\text{cov}(X, Y) < 0$, when $X$ is larger, $Y$ tends to be smaller. $X$ and $Y$ are said to be negatively correlated.

When $\text{cov}(X, Y) = 0$, we say that $X$ and $Y$ are uncorrelated.
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\[ E[X] = 1 \times 0.15 + 2 \times 0.4 + 3 \times 0.45 = 1.9 \]

\[ E[X^2] = 12 \times 0.15 + 2^2 \times 0.4 + 3^2 \times 0.45 = 5.8 \]

\[ E[Y] = 1 \times 0.2 + 2 \times 0.6 + 3 \times 0.2 = 2 \]

\[ E[XY] = 1 \times 0.05 + 2 \times 0.15 + \cdots + 3^2 \times 0.2 = 4.85 \]

\[ \text{cov}(X,Y) = E[XY] - E[X]E[Y] = 1.05 - 1.9 \times 2 = -0.95 \]

\[ \text{var}(X) = E[X^2] - (E[X])^2 = 5.8 - 1.9^2 = 2.19 \]
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**Fact**

(a) var\([X]\) = \text{cov}(X, X)
Properties of Covariance


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Linear Regression: Non-Bayesian

**Definition**
Given the samples \( \{(X_n, Y_n), n = 1, \ldots, N\} \),
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Given the samples \( \{(X_n, Y_n), n = 1, \ldots, N\} \), the Linear Regression of \( Y \) over \( X \) is

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**Why the squares and not the absolute values?**

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**Note:** This is a non-Bayesian formulation: there is no prior.
Linear Least Squares Estimate

Definition

Given two RVs $X$ and $Y$ with known distribution $P[X = x, Y = y]$, the **Linear Least Squares Estimate** of $Y$ given $X$ is

$$\hat{Y} = a + bX =: L[Y | X]$$

where $(a, b)$ minimize $g(a, b) := E[(Y - a - bX)^2]$. Thus, $\hat{Y} = a + bX$ is our guess about $Y$ given $X$.

The squared error is $(Y - \hat{Y})^2$. The LLSE minimizes the expected value of the squared error.

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Note: This is a Bayesian formulation: there is a prior $P[X = x, Y = y]$. 
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Note: This is a **Bayesian** formulation:
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LR: Non-Bayesian or Uniform?

Observe that

$$\sum_{n=1}^{N} (Y_n - a - bX_n)^2 = \mathbb{E}[(Y - a - bX)^2]$$

where one assumes that

$$\mathbb{P}((X, Y) = (X_n, Y_n)) = \frac{1}{N}$$

for \(n = 1, \ldots, N\).

That is, the non-Bayesian LR is equivalent to the Bayesian LLSE that assumes that \((X, Y)\) is uniform on the set of observed samples.

Thus, we can study the two cases LR and LLSE in one shot. However, the interpretations are different!
LR: Non-Bayesian or Uniform?

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\frac{1}{N} \sum_{n=1}^{N} (Y_n - a - bX_n)^2 = E[(Y - a - bX)^2]
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\[(X, Y) = (X_n, Y_n), \text{ w.p. } \frac{1}{N} \text{ for } n = 1, \ldots, N.\]
LR: Non-Bayesian or Uniform?

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That is, the non-Bayesian LR is equivalent to the Bayesian LLSE that assumes that \((X, Y)\) is uniform on the set of observed samples.
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Observe that

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\frac{1}{N} \sum_{n=1}^{N} (Y_n - a - bX_n)^2 = E[(Y - a - bX)^2]
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Next Time.