Today.

Principle of Induction.
Today.

Principle of Induction.

\[ P(0) \land (\forall n \in \mathbb{N}) P(n) \implies P(n + 1) \]
Principle of Induction.

\[ P(0) \land (\forall n \in \mathbb{N}) P(n) \implies P(n+1) \]
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\[ P(0) \land (\forall n \in \mathbb{N}) P(n) \implies P(n + 1) \]

And we get...

...Yes for 0, and we can conclude Yes for 1...

...and we can conclude Yes for 2...

...
Today.

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\[ P(0) \land (\forall n \in \mathbb{N})P(n) \implies P(n + 1) \]

And we get...

\[ (\forall n \in \mathbb{N})P(n). \]
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And we get...

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...Yes for 0,
Principle of Induction.

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And we get...

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...Yes for 0, and we can conclude Yes for 1...

and we can conclude Yes for 2.......
Climb an infinite ladder?
Climb an infinite ladder?
Climb an infinite ladder?

∀ k, P(k) ⇒ P(k+1)

P(0) ⇒ P(1) ⇒ P(2) ⇒ P(3)...

∀ n ∈ N, P(n)
Climb an infinite ladder?

\[ P(0) \]
\[ \forall k, P(k) \implies P(k + 1) \]
Climb an infinite ladder?

∀k, P(k) \implies P(k + 1)

P(0) \implies P(1) \implies P(2)

\ldots

\forall n \in \mathbb{N} \ P(n)
Climb an infinite ladder?

\[ P(0) \]
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\[ P(0) \implies P(1) \implies P(2) \implies P(3) \]
Climb an infinite ladder?

∀ k, P(k) \implies P(k + 1)

P(0) \implies P(1) \implies P(2) \implies P(3) \ldots

Your favorite example of forever.
or the natural numbers...
Climb an infinite ladder?

∀k, \( P(k) \implies P(k + 1) \)

\[ \begin{align*}
P(0) & \implies P(1) \implies P(2) \implies P(3) \ldots
\end{align*} \]
Climb an infinite ladder?

\[ P(0) \]

\[ P(1) \]

\[ P(2) \]

\[ P(3) \]

\[ \forall k, P(k) \implies P(k + 1) \]

\[ P(0) \implies P(1) \implies P(2) \implies P(3) \ldots \]
Climb an infinite ladder?

∀ \( n \in \mathbb{N} \)

\[ P(n) \implies P(n+1) \implies P(n+2) \implies P(n+3) \implies \ldots \]
Climb an infinite ladder?

\[ P(0) \Rightarrow P(1) \Rightarrow P(2) \Rightarrow P(3) \ldots \]

\( (\forall n \in \mathbb{N}) P(n) \)
Climb an infinite ladder?

Your favorite example of forever..
Climb an infinite ladder?

∀ \(n \in \mathbb{N}\), \(P(n) \Rightarrow P(n+1) \Rightarrow P(n+2) \Rightarrow P(n+3) \Rightarrow \ldots \)

Your favorite example of forever..or the natural numbers...
Another Induction Proof.

**Theorem:** For every \( n \in N \), \( n^3 - n \) is divisible by 3. \((3 | (n^3 - n)) \).
Another Induction Proof.

**Theorem:** For every $n \in N$, $n^3 - n$ is divisible by 3. ($3|(n^3 - n)$).

**Proof:**

By induction.

Base Case: $P(0)$ is "$(0^3 - 0)$" is divisible by 3. Yes!

Induction Step: $(\forall k \in N), P(k) \Rightarrow P(k+1)$

Induction Hypothesis: $k^3 - k$ is divisible by 3.

or $k^3 - k = 3q$ for some integer $q$.

$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - (k+1) = k^3 - k + 3k^2 + 3k = (k^3 - k) + 3k^2 + 3k$

Subtract/add $k = 3q + 3(k^2 + k)$

Induction Hyp.

Factor.

$(k+1)^3 - (k+1) = 3(q + k^2 + k)$

(Un)Distributive + over $\times$

$(q + k^2 + k)$ is integer (closed under addition and multiplication).

$\Rightarrow (k+1)^3 - (k+1)$ is divisible by 3.

Thus, $(\forall k \in N) P(k) \Rightarrow P(k+1)$ thus, theorem holds by induction.
Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. ($3 \mid (n^3 - n)$).

Proof: By induction.
Theorem: For every $n \in \mathbb{N}$, $n^3 - n$ is divisible by 3. ($3|\left(n^3 - n\right)$).

Proof: By induction.
Base Case: $P(0)$ is “$(0^3) - 0$” is divisible by 3.
Another Induction Proof.

**Theorem:** For every \( n \in \mathbb{N} \), \( n^3 - n \) is divisible by 3. (\( 3 \mid (n^3 - n) \)).

**Proof:** By induction.
Base Case: \( P(0) \) is \( "(0^3) - 0" \) is divisible by 3. Yes!
Another Induction Proof.

**Theorem:** For every \( n \in N \), \( n^3 - n \) is divisible by 3. \((3 | (n^3 - n))\).

**Proof:** By induction.
Base Case: \( P(0) \) is “\((0^3) - 0\)” is divisible by 3. Yes!
Induction Step: \((\forall k \in N), P(k) \implies P(k + 1)\)
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**Theorem:** For every \( n \in \mathbb{N} \), \( n^3 - n \) is divisible by 3. \( (3 | (n^3 - n)) \).

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Base Case: \( P(0) \) is “\((0^3) - 0\)” is divisible by 3. Yes!
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or \( k^3 - k = 3q \) for some integer \( q \).
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**Theorem:** For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n))$.

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Base Case: $P(0)$ is “$(0^3) - 0$” is divisible by 3. Yes!
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**Theorem:** For every \( n \in \mathbb{N}, \ n^3 - n \) is divisible by 3. \((3 | (n^3 - n))\).

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\((k + 1)^3 - (k + 1)\)
Another Induction Proof.

**Theorem:** For every \( n \in \mathbb{N} \), \( n^3 - n \) is divisible by 3. (\( 3 | (n^3 - n) \)).

**Proof:** By induction.

Base Case: \( P(0) \) is “\( (0^3) - 0 \)” is divisible by 3. Yes!

Induction Step: \( (\forall k \in \mathbb{N}), P(k) \implies P(k + 1) \)

Induction Hypothesis: \( k^3 - k \) is divisible by 3.

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\[
(k + 1)^3 - (k + 1) = k^3 + 3k^2 + 3k + 1 - (k + 1)
\]
Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. ($3 | (n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is “$(0^3) - 0$” is divisible by 3. Yes!
Induction Step: $(\forall k \in N), P(k) \implies P(k + 1)$
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$$(k + 1)^3 - (k + 1) = k^3 + 3k^2 + 3k + 1 - (k + 1)$$

$$= k^3 + 3k^2 + 2k$$
Another Induction Proof.

**Theorem:** For every \( n \in N \), \( n^3 - n \) is divisible by 3. \((3|(n^3 - n))\).

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(k + 1)^3 - (k + 1) = k^3 + 3k^2 + 3k + 1 - (k + 1)
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= (k^3 - k) + 3k^2 + 3k
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**Theorem:** For every $n \in \mathbb{N}$, $n^3 - n$ is divisible by 3. ($3|(n^3 - n)$).

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(k + 1)^3 - (k + 1) = k^3 + 3k^2 + 3k + 1 - (k + 1) \\
= k^3 + 3k^2 + 2k \\
= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k \\
= 3q + 3(k^2 + k)
$$
Another Induction Proof.

**Theorem:** For every \( n \in N \), \( n^3 - n \) is divisible by 3. (3|\((n^3 - n)\)).

**Proof:** By induction.

Base Case: \( P(0) \) is "\((0^3) - 0\)" is divisible by 3. Yes!

Induction Step: \((\forall k \in N), P(k) \implies P(k + 1)\)

Induction Hypothesis: \( k^3 - k \) is divisible by 3.  
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(k + 1)^3 - (k + 1) = k^3 + 3k^2 + 3k + 1 - (k + 1)
= k^3 + 3k^2 + 2k
= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k
= 3q + 3(k^2 + k) \quad \text{Induction Hyp.}
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Another Induction Proof.

**Theorem:** For every $n \in N$, $n^3 - n$ is divisible by 3. ($3|(n^3 - n)$).

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$$(k + 1)^3 - (k + 1) = k^3 + 3k^2 + 3k + 1 - (k + 1)$$
$$= k^3 + 3k^2 + 2k$$
$$= (k^3 - k) + 3k^2 + 3k$$
$$= 3q + 3(k^2 + k)$$

Induction Hyp. Factor.
Another Induction Proof.

**Theorem:** For every \( n \in N \), \( n^3 - n \) is divisible by 3. \((3|(n^3 - n) )\).

**Proof:** By induction.
Base Case: \( P(0) \) is “\((0^3) - 0\)” is divisible by 3. Yes!
Induction Step: \( (\forall k \in N), P(k) \implies P(k + 1) \)
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**Theorem:** For every \( n \in \mathbb{N} \), \( n^3 - n \) is divisible by 3. \((3 \mid (n^3 - n))\).

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= 3q + 3(k^2 + k) \quad \text{Induction Hyp. Factor.}
\]
\[
= 3(q + k^2 + k) \quad \text{(Un)Distributive + over } \times
\]
Theorem: For every \( n \in N \), \( n^3 - n \) is divisible by 3. \((3 | (n^3 - n))\).

Proof: By induction.

Base Case: \( P(0) \) is \((0^3) - 0 \) is divisible by 3. Yes!

Induction Step: \( (\forall k \in N), P(k) \implies P(k + 1) \)

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Or \( (k + 1)^3 - (k + 1) = 3(q + k^2 + k) \).
Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. ($3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is “$(0^3) - 0$” is divisible by 3. Yes!
Induction Step: $(\forall k \in N), P(k) \implies P(k + 1)$
Induction Hypothesis: $k^3 - k$ is divisible by 3.
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$$\begin{align*}
(k + 1)^3 - (k + 1) &= k^3 + 3k^2 + 3k + 1 - (k + 1) \\
&= k^3 + 3k^2 + 2k \\
&= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k \\
&= 3q + 3(k^2 + k) \quad \text{Induction Hyp. Factor.} \\
&= 3(q + k^2 + k) \quad \text{(Un)Distributive + over } \times
\end{align*}$$

Or $(k + 1)^3 - (k + 1) = 3(q + k^2 + k)$.

$(q + k^2 + k)$ is integer (closed under addition and multiplication).
**Theorem:** For every \( n \in N \), \( n^3 - n \) is divisible by 3. \((3|(n^3 - n))\).

**Proof:** By induction.

Base Case: \( P(0) \) is “\((0^3) - 0\)” is divisible by 3. Yes!

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= 3(q + k^2 + k) \quad \text{(Un)Distributive + over } \times
\]

Or \( (k + 1)^3 - (k + 1) = 3(q + k^2 + k). \)

\( (q + k^2 + k) \) is integer (closed under addition and multiplication).

\( \implies (k + 1)^3 - (k + 1) \) is divisible by 3.
Another Induction Proof.

**Theorem:** For every $n \in \mathbb{N}$, $n^3 - n$ is divisible by 3. ($3 | (n^3 - n)$).

**Proof:** By induction.  
Base Case: $P(0)$ is “$(0^3) - 0$” is divisible by 3. Yes!  
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$$(k + 1)^3 - (k + 1) = k^3 + 3k^2 + 3k + 1 - (k + 1)$$
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$$= 3q + 3(k^2 + k)$$ Induction Hyp. Factor.
$$= 3(q + k^2 + k)$$ (Un)Distributive + over $\times$

Or $(k + 1)^3 - (k + 1) = 3(q + k^2 + k)$.

$(q + k^2 + k)$ is integer (closed under addition and multiplication).

$$\implies (k + 1)^3 - (k + 1)$$ is divisible by 3.

Thus, $(\forall k \in \mathbb{N}) P(k) \implies P(k + 1)$
Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3 | (n^3 - n))$.

Proof: By induction.

Base Case: $P(0)$ is “$(0^3) - 0$” is divisible by 3. Yes!

Induction Step: $(\forall k \in N), P(k) \implies P(k + 1)$

Induction Hypothesis: $k^3 - k$ is divisible by 3.

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$$(k + 1)^3 - (k + 1) = k^3 + 3k^2 + 3k + 1 - (k + 1)$$

$$= k^3 + 3k^2 + 2k$$

$$= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k$$

$$= 3q + 3(k^2 + k) \quad \text{Induction Hyp.} \quad \text{Factor.}$$

$$= 3(q + k^2 + k) \quad \text{(Un)Distributive + over} \times$$

Or $(k + 1)^3 - (k + 1) = 3(q + k^2 + k)$.

$(q + k^2 + k)$ is integer (closed under addition and multiplication).

$\implies (k + 1)^3 - (k + 1)$ is divisible by 3.

Thus, $(\forall k \in N)P(k) \implies P(k + 1)$

Thus, theorem holds by induction.
Another Induction Proof.

**Theorem:** For every $n \in N$, $n^3 - n$ is divisible by 3. ($3 | (n^3 - n)$).

**Proof:** By induction.

Base Case: $P(0)$ is “$(0^3) - 0$” is divisible by 3. Yes!

Induction Step: $(\forall k \in N), \ P(k) \implies P(k+1)$

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$$= 3(q + k^2 + k) \quad \text{(Un)Distributive + over } \times$$

Or $(k + 1)^3 - (k + 1) = 3(q + k^2 + k)$.

$(q + k^2 + k)$ is integer (closed under addition and multiplication).

$
\implies (k + 1)^3 - (k + 1)$ is divisible by 3.

Thus, $(\forall k \in N) P(k) \implies P(k+1)$

Thus, theorem holds by induction.

$\square$
**Four Color Theorem.**

**Theorem:** Any map can be colored so that those regions that share an edge have different colors.

![Map of the United States colored using the Four Color Theorem](image-url)
**Four Color Theorem.**

**Theorem:** Any map can be colored so that those regions that share an edge have different colors.

Check Out: “Four corners”.

![Map of the United States with states colored according to the Four Color Theorem.](image-url)
**Four Color Theorem.**

**Theorem:** Any map can be colored so that those regions that share an edge have different colors.

Check Out: “Four corners”. States connected at a point, can have same color.
Four Color Theorem.

**Theorem:** Any map can be colored so that those regions that share an edge have different colors.

Check Out: “Four corners”. States connected at a point, can have same color. (Couldn’t find a map where they did though.)
Four Color Theorem.

**Theorem:** Any map can be colored so that those regions that share an edge have different colors.

Check Out: “Four corners”. States connected at a point, can have same color. (Couldn’t find a map where they did though.)

Quick Test: Which states?
Four Color Theorem.

**Theorem:** Any map can be colored so that those regions that share an edge have different colors.

Check Out: “Four corners”.
States connected at a point, can have same color. (Couldn’t find a map where they did though.)

Quick Test: Which states? Utah.
Four Color Theorem.

**Theorem:** Any map can be colored so that those regions that share an edge have different colors.

Check Out: “Four corners”.
States connected at a point, can have same color.
(Couldn’t find a map where they did though.)

Four Color Theorem.

**Theorem:** Any map can be colored so that those regions that share an edge have different colors.

Check Out: “Four corners”.
States connected at a point, can have same color.
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Four Color Theorem.

Theorem: Any map can be colored so that those regions that share an edge have different colors.

Check Out: “Four corners”.
States connected at a point, can have same color.
(Couldn’t find a map where they did though.)

Two color theorem: example.

Any map formed by dividing the plane into regions by drawing straight lines can be properly colored with two colors.
Two color theorem: example.

Any map formed by dividing the plane into regions by drawing straight lines can be properly colored with two colors.
Two color theorem: example.

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Two color theorem: example.

Any map formed by dividing the plane into regions by drawing straight lines can be properly colored with two colors.

Fact: Swapping red and blue gives another valid colors.
Two color theorem: example.

Any map formed by dividing the plane into regions by drawing straight lines can be properly colored with two colors.

Fact: Swapping red and blue gives another valid colors.
Base Case.
Two color theorem: proof illustration.

Base Case.
Two color theorem: proof illustration.

1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line. (Fixes conflicts along line, and makes no new ones.)

Algorithm gives $P(k) \Rightarrow P(k+1)$.
Two color theorem: proof illustration.

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[Diagram of two color theorem illustration]
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Algorithm gives $P(k) \implies P(k + 1)$. 

\[ \square \]
Strengthening Induction Hypothesis.

**Theorem:** The sum of the first $n$ odd numbers is a perfect square.
**Theorem:** The sum of the first \( n \) odd numbers is a perfect square.

\[
k\text{th odd number is } 2(k - 1) + 1.
\]
Theorem: The sum of the first $n$ odd numbers is a perfect square.

$k$th odd number is $2(k - 1) + 1$.

Base Case 1 (first odd number) is $1^2$. 

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Strengthening Induction Hypothesis.

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**Induction Step** 1. The $(k + 1)$st odd number is $2k + 1$. 
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Induction Step 1. The $(k + 1)$st odd number is $2k + 1$.
2. Sum of the first $k + 1$ odds is $a^2 + 2k + 1$. 

Strenthening Induction Hypothesis.
Strengthening Induction Hypothesis.

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**Induction Hypothesis** Sum of first $k$ odds is perfect square $a^2 = k^2$.

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Induction Step

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\[ k^2 + 2k + 1 = (k + 1)^2 \]

... \( P(k+1)! \)
Strengthening Induction Hypothesis.

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   ... $P(k+1)!$
Tiling Cory Hall Courtyard.

Use these $L$-tiles.

To Tile this $4 \times 4$ courtyard.
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\[ A \quad C \quad \quad B \quad D \quad E \]
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To Tile this $4 \times 4$ courtyard.

Alright!
Tiling Cory Hall Courtyard.

Use these $L$-tiles.

To Tile this $4 \times 4$ courtyard.

Alright!

Tiled $4 \times 4$ square with $2 \times 2$ $L$-tiles.
Tiling Cory Hall Courtyard.

Use these $L$-tiles.

To Tile this $4 \times 4$ courtyard.

Alright!

Tiled $4 \times 4$ square with $2 \times 2$ $L$-tiles.
with a center hole.
Tiling Cory Hall Courtyard.

To Tile this $4 \times 4$ courtyard.

Use these $L$-tiles.

Alright!

**Tiled $4 \times 4$ square with $2 \times 2$ $L$-tiles.**

with a center hole.

Can we tile any $2^n \times 2^n$ with $L$-tiles (with a hole)
Tiling Cory Hall Courtyard.

Use these $L$-tiles.

To Tile this $4 \times 4$ courtyard.

Alright!

Tiled $4 \times 4$ square with $2 \times 2$ $L$-tiles.
with a center hole.

Can we tile any $2^n \times 2^n$ with $L$-tiles (with a hole) for every $n$!
Hole have to be there? Maybe just one?

**Theorem:** Any tiling of $2^n \times 2^n$ square has to have one hole.
Hole have to be there? Maybe just one?

**Theorem:** Any tiling of $2^n \times 2^n$ square has to have one hole.

**Proof:** The remainder of $2^{2n}$ divided by 3 is 1.
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**Theorem:** Any tiling of $2^n \times 2^n$ square has to have one hole.

**Proof:** The remainder of $2^{2n}$ divided by 3 is 1.

Base case: true for $k = 0$. 
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**Theorem:** Any tiling of $2^n \times 2^n$ square has to have one hole.

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Ind Hyp: $2^{2k} = 3a + 1$ for integer $a$. 

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$$2^{2(k+1)}$$
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Base case: true for $k = 0$. $2^0 = 1$
Ind Hyp: $2^{2k} = 3a + 1$ for integer $a$.

$$2^{2(k+1)} = 2^{2k} \times 2^2$$
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\[
2^{2(k+1)} = 2^{2k} \cdot 2^2 = 4 \cdot 2^{2k}
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$$
2^{2(k+1)} = 2^{2k} \times 2^2 \\
= 4 \times 2^{2k} \\
= 4 \times (3a + 1)
$$
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2^{2(k+1)} = 2^{2k} \times 2^2
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= 4 \times 2^{2k}
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\[
= 12a + 3 + 1
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2^{2(k+1)} = 2^{2k} \times 2^2 \\
= 4 \times 2^{2k} \\
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= 3(4a + 1) + 1
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\begin{align*}
2^{2(k+1)} & = 2^{2k} \times 2^2 \\
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$a$ integer
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2^{2(k+1)} &= 2^{2k} \cdot 2^2 \\
&= 4 \cdot 2^{2k} \\
&= 4 \cdot (3a + 1) \\
&= 12a + 3 + 1 \\
&= 3(4a + 1) + 1
\end{align*}
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$a$ integer $\implies (4a + 1)$ is an integer.
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\[
2^{2(k+1)} = 2^{2k} \times 2^2 = 4 \times 2^{2k} = 4 \times (3a + 1) = 12a + 3 + 1 = 3(4a + 1) + 1
\]

$a$ integer $\implies (4a + 1)$ is an integer.
Hole in center?

**Theorem:** Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

**Proof:**
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Proof:

Base case: A single tile works fine.
Hole in center?

**Theorem:** Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

**Proof:**

Base case: A single tile works fine.
   The hole is adjacent to the center of the $2 \times 2$ square.
Hole in center?

**Theorem:** Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

**Proof:**

Base case: A single tile works fine.

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Induction Hypothesis:
Hole in center?

**Theorem:** Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

**Proof:**

Base case: A single tile works fine.
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Induction Hypothesis:
Any $2^n \times 2^n$ square can be tiled with a hole at the center.
Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

Proof:

Base case: A single tile works fine.
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Any $2^n \times 2^n$ square can be tiled with a hole at the center.
Hole in center?

**Theorem:** Can tile the \( 2^n \times 2^n \) square to leave a hole adjacent to the center.

**Proof:**

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Induction Hypothesis:
Any \( 2^n \times 2^n \) square can be tiled with a hole at the center.

\[
2^{n+1}
\]

\[
\begin{array}{cc}
\text{hole} & \text{hole} \\
\text{hole} & \text{hole}
\end{array}
\]
Hole can be anywhere!

**Theorem:** Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere*. 

Better theorem... better induction hypothesis!

Base case: Sure. A tile is fine. Flipping the orientation can leave hole anywhere.

Induction Hypothesis: "Any $2^n \times 2^n$ square can be tiled with a hole anywhere."

Consider $2^{n+1} \times 2^{n+1}$ square. Use induction hypothesis in each. Use L-tile and ... we are done.
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Use induction hypothesis in each.

Use L-tile and ...
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Use induction hypothesis in each.

Use L-tile and ... we are done.
Strong Induction.

**Theorem:** Every natural number \( n > 1 \) can be written as a (possibly trivial) product of primes.
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**Theorem:** Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

**Definition:** A prime $n$ has exactly 2 factors 1 and $n$. 
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Strong Induction.

**Theorem:** Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

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**Base Case:** $n = 2$.

**Induction Step:**

$P(n) =$ “$n$ can be written as a product of primes.”

Strong Induction Principle:

If $P(0)$ and $(\forall k \in \mathbb{N})(P(0) \land \ldots \land P(k)) \Rightarrow P(k + 1)$,

then $(\forall k \in \mathbb{N})(P(k))$.
Strong Induction.

**Theorem:** Every natural number \( n > 1 \) can be written as a (possibly trivial) product of primes.

**Definition:** A prime \( n \) has exactly 2 factors 1 and \( n \).

**Base Case:** \( n = 2 \).

**Induction Step:**

\( P(n) = \) “\( n \) can be written as a product of primes. “

Either \( n + 1 \) is a prime
Strong Induction.

**Theorem:** Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

**Definition:** A prime $n$ has exactly 2 factors 1 and $n$.

**Base Case:** $n = 2$.

**Induction Step:**

$P(n) =$ “$n$ can be written as a product of primes. “

Either $n + 1$ is a prime or $n + 1 = a \cdot b$ where $1 < a, b < n + 1$. 
Strong Induction.

**Theorem:** Every natural number \( n > 1 \) can be written as a (possibly trivial) product of primes.

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\( P(n) \) = “\( n \) can be written as a product of primes. “

Either \( n + 1 \) is a prime or \( n + 1 = a \cdot b \) where \( 1 < a, b < n + 1 \).

\( P(n) \) says nothing about \( a, b \)!
**Strong Induction.**

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---

**Strong Induction Principle:** If \( P(0) \) and

\[
(\forall k \in N)((P(0) \land \ldots \land P(k)) \implies P(k + 1)),
\]

then \( (\forall k \in N)(P(k)) \).
**Strong Induction.**

**Theorem:** Every natural number \( n > 1 \) can be written as a (possibly trivial) product of primes.

**Definition:** A prime \( n \) has exactly 2 factors 1 and \( n \).

**Base Case:** \( n = 2 \).

**Induction Step:**

\( P(n) = \text{"n can be written as a product of primes."} \)

Either \( n + 1 \) is a prime or \( n + 1 = a \cdot b \) where \( 1 < a, b < n + 1 \).

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---

**Strong Induction Principle:** If \( P(0) \) and

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(\forall k \in \mathbb{N})((P(0) \land \ldots \land P(k)) \implies P(k+1)),
\]

then \( (\forall k \in \mathbb{N})(P(k)) \).

\[
P(0) \implies P(1) \implies P(2) \implies P(3) \implies \ldots
\]
Strong Induction.

**Theorem:** Every natural number \( n > 1 \) can be written as a (possibly trivial) product of primes.

**Definition:** A prime \( n \) has exactly 2 factors 1 and \( n \).

**Base Case:** \( n = 2 \).

**Induction Step:**

\( P(n) = \text{"n can be written as a product of primes. "} \)

Either \( n + 1 \) is a prime or \( n + 1 = a \cdot b \) where \( 1 < a, b < n + 1 \).

\( P(n) \) says nothing about \( a, b \)!

---

**Strong Induction Principle:** If \( P(0) \) and

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(\forall k \in N)((P(0) \land \ldots \land P(k)) \implies P(k + 1)),
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then \( (\forall k \in N)(P(k)) \).

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Strong induction hypothesis: “$a$ and $b$ are products of primes”
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Strong induction hypothesis: “\( a \) and \( b \) are products of primes”

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\implies \text{“} n + 1 = a \cdot b = \text{(factorization of } a)\text{(factorization of } b)\text{”}
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\( n + 1 \) can be written as the product of the prime factors!
Strong Induction.

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Strong induction hypothesis: “$a$ and $b$ are products of primes”

$\implies$ “$n + 1 = a \cdot b =$ (factorization of $a$)(factorization of $b$)”

$n + 1$ can be written as the product of the prime factors!
Induction $\implies$ Strong Induction.

Let $Q(k) = P(0) \land P(1) \cdots P(k)$. 
Induction $\Rightarrow$ Strong Induction.

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Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.
Well Ordering Principle and Induction.

If $(\forall n) P(n)$ is not true, then $(\exists n) \neg P(n)$.

Consider smallest $m$, with $\neg P(m)$, $m \geq 0$
Well Ordering Principle and Induction.

If \((\forall n) P(n)\) is not true, then \((\exists n) \neg P(n)\).
Consider smallest \(m\), with \(\neg P(m), m \geq 0\)
\(P(m-1) \implies P(m)\) must be false (assuming \(P(0)\) holds.)
Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\lnot P(n)$.

Consider smallest $m$, with $\lnot P(m)$, $m \geq 0$

$P(m - 1) \implies P(m)$ must be false (assuming $P(0)$ holds.)

This is a proof of the induction principle!

I.e.,

$$(\lnot \forall n)P(n) \implies ((\exists n)\lnot (P(n-1) \implies P(n))).$$
Well Ordering Principle and Induction.

If \((\forall n)P(n)\) is not true, then \((\exists n)\neg P(n)\).
Consider smallest \(m\), with \(\neg P(m)\), \(m \geq 0\)
\(P(m-1) \implies P(m)\) must be false (assuming \(P(0)\) holds.)
This is a proof of the induction principle!
I.e.,
\[
(\neg \forall n)P(n) \implies ((\exists n)(\neg (P(n-1) \implies P(n)))).
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(Contrapositive of Induction principle (assuming \(P(0)\))
Well Ordering Principle and Induction.

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It assumes that there is a smallest \(m\) where \(P(m)\) does not hold.
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The **Well ordering principle** states that for any subset of the natural numbers there is a smallest element.
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Smallest may not be what you expect: the well ordering principal holds for rationals but with different ordering!!
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The **Well ordering principle** states that for any subset of the natural numbers there is a smallest element.

Smallest may not be what you expect: the well ordering principal holds for rationals but with different ordering!!

E.g. Reduced form is “smallest” representation of a rational number $a/b$. 
Well ordering principle.

Thm: All natural numbers are interesting.
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0 is interesting...
Well ordering principle.

Thm: All natural numbers are interesting.
0 is interesting...
Let $n$ be the first uninteresting number.
Well ordering principle.

Thm: All natural numbers are interesting.

0 is interesting...

Let \( n \) be the first uninteresting number.

But \( n - 1 \) is interesting and \( n \) is uninteresting,
Thm: All natural numbers are interesting.

0 is interesting...

Let $n$ be the first uninteresting number.
  But $n - 1$ is interesting and $n$ is uninteresting, so this is the first uninteresting number.
Well ordering principle.

Thm: All natural numbers are interesting.

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Let $n$ be the first uninteresting number.
   But $n – 1$ is interesting and $n$ is uninteresting,
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Thm: All natural numbers are interesting.

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Let \( n \) be the first uninteresting number.
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Thus, there is no smallest uninteresting natural number.
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Thm: All natural numbers are interesting.

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  But \( n - 1 \) is interesting and \( n \) is uninteresting,
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  But this is interesting.
Thus, there is no smallest uninteresting natural number.

Thus: All natural numbers are interesting.
Def: A **round robin tournament on** $n$ **players**: every player $p$ plays every other player $q$, and either $p \rightarrow q$ ($p$ beats $q$) or $q \rightarrow p$ ($q$ beats $p$.)
Tournaments have short cycles

Def: A **round robin tournament on** $n$ **players**: every player $p$ plays every other player $q$, and either $p \rightarrow q$ ($p$ beats $q$) or $q \rightarrow p$ ($q$ beats $p$.)

Def: A **cycle**: a sequence of $p_1, \ldots, p_k$, $p_i \rightarrow p_{i+1}$ and $p_k \rightarrow p_1$. 
Tournaments have short cycles

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![Diagram of a tournament with cycles](image)
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Theorem: Any tournament that has a cycle has a cycle of length 3.
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Tournament has a cycle of length 3 if at all.
Tournament has a cycle of length 3 if at all.

Assume the smallest cycle is of length $k$. 

Case 1: Of length 3. Done.

Case 2: Of length larger than 3.

$p_1 \rightarrow p_2 \rightarrow p_3 \rightarrow \cdots \rightarrow p_k \rightarrow \cdots \rightarrow \cdots = \cdots \rightarrow p_1$ \Rightarrow 3 cycle

Contradiction.

$p_1 \rightarrow p_3$ \Rightarrow $k-1$ length cycle!

Contradiction!
Tournament has a cycle of length 3 if at all.

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Tournament has a cycle of length 3 if at all.

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```
$p_1 \rightarrow p_3 \rightarrow p_1$ $\implies$ 3 cycle
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\[ p_1 \rightarrow p_3 = \Rightarrow 3 \text{ cycle} \]

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```
\[
p_1 \rightarrow p_3 \quad \Rightarrow \quad p_3 \rightarrow p_1 \quad \Rightarrow \quad 3 \text{ cycle}
\]
```

Contradiction.

```
\[
p_1 \rightarrow p_3 \quad \Rightarrow \quad k - 1 \text{ length cycle!}
\]
```

Contradiction!
Tournaments have long paths.

Def: A round robin tournament on $n$ players: every player $p$ plays every other player $q$, and either $p \rightarrow q$ ($p$ beats $q$) or $q \rightarrow q$ ($q$ beats $q$.)
Tournaments have long paths.

Def: A round robin tournament on $n$ players: every player $p$ plays every other player $q$, and either $p \rightarrow q$ ($p$ beats $q$) or $q \rightarrow q$ ($q$ beats $q$.)

Def: A Hamiltonian path: a sequence
Tournaments have long paths.

**Def:** A round robin tournament on \(n\) players: every player \(p\) plays every other player \(q\), and either \(p \rightarrow q\) (\(p\) beats \(q\)) or \(q \rightarrow q\) (\(q\) beats \(q\)).

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Tournaments have long paths.

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Def: A Hamiltonian path: a sequence \( p_1, \ldots, p_n \), \( (\forall i, 0 \leq i < n) \) \( p_i \to p_{i+1} \).

Base: True for two vertices.
Tournaments have long paths.

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\[ p_1, \ldots, p_n, \quad (\forall i, 0 \leq i < n) \quad p_i \rightarrow p_{i+1}. \]

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(Also for one, but two is more useful as base case!)
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Tournament on \( n + 1 \) people,
Tournaments have long paths.

**Def:** A round robin tournament on $n$ players: every player $p$ plays every other player $q$, and either $p \to q$ ($p$ beats $q$) or $q \to q$ ($q$ beats $q$.)

**Def:** A Hamiltonian path: a sequence $p_1, \ldots, p_n$, ($\forall i, 0 \leq i < n$) $p_i \to p_{i+1}$.

Base: True for two vertices. (Also for one, but two is more useful as base case!)

Tournament on $n+1$ people,
Remove arbitrary person
Tournaments have long paths.

**Def:** A **round robin tournament on** $n$ **players:** every player $p$ plays every other player $q$, and either $p \rightarrow q$ ($p$ beats $q$) or $q \rightarrow q$ ($q$ beats $q$.)

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**Tournament on** $n + 1$ **people,**

Remove arbitrary person $\rightarrow$ yield tournament on $n - 1$ people.
Tournaments have long paths.

**Def:** A **round robin tournament on** \( n \) **players:** every player \( p \) plays every other player \( q \), and either \( p \rightarrow q \) (\( p \) beats \( q \)) or \( q \rightarrow q \) (\( q \) beats \( q \)).

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Tournament on \( n + 1 \) people,
Remove arbitrary person \( \rightarrow \) yield tournament on \( n - 1 \) people.
(RESULT specified for each remaining pair from original tournament.)
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**Def:** A *round robin tournament on n players*: every player \( p \) plays every other player \( q \), and either \( p \rightarrow q \) (\( p \) beats \( q \)) or \( q \rightarrow q \) (\( q \) beats \( q \)).

**Def:** A *Hamiltonian path*: a sequence
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Tournament on \( n + 1 \) people,
Remove arbitrary person \( \rightarrow \) yield tournament on \( n – 1 \) people.
(Result specified for each remaining pair from original tournament.)

By induction hypothesis: There is a sequence \( p_1, \ldots, p_n \).
Tournaments have long paths.

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If \( p \) is big winner, put at beginning.
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Horses of the same color...

**Theorem:** All horses have the same color.
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Base Case: $P(1)$ - trivially true.
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As we will see, it is more subtle to catch errors in proofs of correct theorems!!
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New Base Case: $P(2)$: there are two horses with same color.

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As we will see, it is more subtle to catch errors in proofs of correct theorems!!
Strong Induction and Recursion.

Thm: For every natural number \( n \geq 12 \), \( n = 4x + 5y \).
Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let’s write some code!

```python
def find_x_y(n):
    if (n==12): return (3,0)
    elif (n==13): return(2,1)
    elif (n==14): return(1,2)
    elif (n==15): return(0,3)
    else:
        (x',y') = find_x_y(n-4)
        return(x'+1,y')
```

Base cases: $P(12), P(13), P(14), P(15)$.

Yes.

Strong Induction step: Recursive call is correct: $P(n-4) \Rightarrow P(n)$.

$n - 4 = 4x' + 5y' \Rightarrow n = 4(x' + 1) + 5y'$. 

Slight differences: showed for all $n \geq 16$ that $\land \ i = 4P(i) \Rightarrow P(n)$. 

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Strong Induction step:
Recursive call is correct: $P(n-4)$
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```

Base cases: \( P(12) \), \( P(13) \), \( P(14) \), \( P(15) \). Yes.

Strong Induction step:
Recursive call is correct: \( P(n-4) \implies P(n) \).
\[
n - 4 = 4x' + 5y' \implies n = 4(x' + 1) + 5(y')
\]
Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let’s write some code!

def find-x-y(n):
    if (n==12) return (3,0)
    elif (n==13): return(2,1)
    elif (n==14): return(1,2)
    elif (n==15): return(0,3)
    else:
        (x',y') = find-x-y(n-4)
        return(x'+1,y')


Strong Induction step:
   Recursive call is correct: $P(n-4) \implies P(n)$.
   $n - 4 = 4x' + 5y' \implies n = 4(x' + 1) + 5(y')$

Slight differences: showed for all $n \geq 16$ that $\land_{i=4}^{n-1} P(i) \implies P(n)$.
Sad Islanders...

Island with 100 possibly blue-eyed and green-eyed inhabitants.
Sad Islanders...

Island with 100 possibly blue-eyed and green-eyed inhabitants.
Any islander who knows they have green eyes must kill themselves that day.
Sad Islanders...

Island with 100 possibly blue-eyed and green-eyed inhabitants.
Any islander who knows they have green eyes must kill themselves that day.
No islander knows their own eye color, but knows everyone else's.

Visitor: "I see someone has green eyes."
Result:
First rule of island: Don't talk about eye color!
On day 100, they all kill themselves.
Why?
Sad Islanders...

Island with 100 possibly blue-eyed and green-eyed inhabitants. Any islander who knows they have green eyes must kill themselves that day. No islander knows their own eye color, but knows everyone else's. All islanders have green eyes!
Sad Islanders...

Island with 100 possibly blue-eyed and green-eyed inhabitants.
Any islander who knows they have green eyes must kill themselves that day.
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Island with 100 possibly blue-eyed and green-eyed inhabitants. Any islander who knows they have green eyes must kill themselves that day.

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Why?
They know induction.

Thm: If there are \( n \) villagers with green eyes they kill themselves on day \( n \).
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**Proof:**
Base: $n = 1$. Person with green eyes kills themselves on day 1.
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**Proof:**
Base: $n = 1$. Person with green eyes kills themselves on day 1.

Induction hypothesis:
If there were $n$ people with green eyes, they would pass away on day $n$. 

But they didn't kill themselves. So there must be $n + 1$ people with green eyes. One of them, is me. Sad.

Wait! Visitor added no information.
They know induction.

Thm: If there are \( n \) villagers with green eyes they kill themselves on day \( n \).

**Proof:**
Base: \( n = 1 \). Person with green eyes kills themselves on day 1.

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Wait! Visitor added no information.
Using knowledge about what other people’s knowledge (your eye color) is.
Common Knowledge.

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Using knowledge about what other people’s knowledge (your eye color) is.
On day 1, everyone knows everyone sees more than zero.
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On day 99, no one sees 98
Common Knowledge.

Using knowledge about what other people’s knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.
On day 2, everyone knows everyone sees more than one.
...
On day 99, no one sees 98 since everyone knows everyone else does not see 97...
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On day 1, everyone knows everyone sees more than zero.
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On day 100,
Common Knowledge.

Using knowledge about what other people’s knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.

On day 2, everyone knows everyone sees more than one.

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On day 100, ...uh oh!
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On day 1, everyone knows everyone sees more than zero.

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On day 99, no one sees 98 since everyone knows everyone else does not see 97...

On day 100, ...uh oh!

Another example:
Common Knowledge.

Using knowledge about what other people’s knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.
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On day 99, no one sees 98 since everyone knows everyone else does not see 97...
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Another example:
Emperor’s new clothes!
Common Knowledge.

Using knowledge about what other people’s knowledge (your eye color) is.
On day 1, everyone knows everyone sees more than zero.
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On day 100, ...uh oh!

Another example:
Emperor’s new clothes!
No one knows other people see that he has no clothes.
Common Knowledge.

Using knowledge about what other people’s knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.
On day 2, everyone knows everyone sees more than one.

... 

On day 99, no one sees 98 since everyone knows everyone else does not see 97...

On day 100, ...uh oh!

Another example:
Emperor’s new clothes!
- No one knows other people see that he has no clothes.
- Until kid points it out.
Summary: principle of induction.

Today: More induction.
Summary: principle of induction.

Today: More induction.

\( (P(0) \land \forall k \in \mathbb{N}(P(k) \Rightarrow P(k+1))) \Rightarrow \forall n \in \mathbb{N}(P(n)) \)

Statement to prove:

Base Case: Prove \( P(n_0) \).

Ind. Step: Prove.

For all values, \( n \geq n_0 \), \( P(n) = \Rightarrow P(n+1) \).

Statement is proven!

Strong Induction:

\( (P(0) \land \forall n \in \mathbb{N}(P(n)) = \Rightarrow P(n+1))) \Rightarrow \forall n \in \mathbb{N}(P(n)) \)

Also Today: strengthened induction hypothesis.

Strengthen theorem statement.

Sum of first \( n \) odds is \( n^2 \).

Hole anywhere.

Not same as strong induction.

E.g., used in product of primes proof.

Induction ≡ Recursion.
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\((P(0) \land ((\forall k \in N)(P(k) \implies P(k + 1))))\)
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**Strengthen theorem statement.**

Sum of first \(n\) odds is \(n^2\).
Summary: principle of induction.

Today: More induction.

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**Strengthen theorem statement.**
- Sum of first \(n\) odds is \(n^2\).
- Hole anywhere.
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Today: More induction.

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Ind. Step: Prove. For all values, \( n \geq n_0 \), \( P(n) \implies P(n+1) \).
Summary: principle of induction.

\((P(0) \land ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))\)

Variations:
\((P(0) \land ((\forall n \in N)(P(n) \implies P(n+1)))) \implies (\forall n \in N)(P(n))\)
\((P(1) \land ((\forall n \in N)((n \geq 1) \land P(n)) \implies P(n+1)))) \implies (\forall n \in N)((n \geq 1) \implies P(n))\)

Statement to prove: \(P(n)\) for \(n\) starting from \(n_0\)
Base Case: Prove \(P(n_0)\).
Ind. Step: Prove. For all values, \(n \geq n_0\), \(P(n) \implies P(n+1)\).
Statement is proven!