Today.

Planar Five Color theorem.
Types of graphs.
Complete Graphs.
Trees.
Hypercubes.

Graph Coloring.

Given $G = (V, E)$, a coloring of a $G$ assigns colors to vertices $V$ where for each edge the endpoints have different colors.

Notice that the last one, has one three colors.
Fewer colors than number of vertices.
Fewer colors than max degree node.
Interesting things to do. Algorithm!

Planar graphs and maps.

Planar graph coloring ≡ map coloring.
Four color theorem is about planar graphs!

Six color theorem.

Theorem: Every planar graph can be colored with six colors.
Proof:
Recall: $e \leq 3v - 6$ for any planar graph where $v > 2$.
From Euler's Formula.
Total degree: $2e$
Average degree: $\leq \frac{2e}{v} \leq \frac{2(3v-6)}{v} \leq 6 - \frac{12}{v}$.
There exists a vertex with degree < 6 or at most 5.
Remove vertex $v$ of degree < 6 or at most 5.
Inductively color remaining graph.
Color is available for $v$ since only five neighbors...
and only five colors are used.

Five color theorem: preliminary.

Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.

Proof:
Again with the degree 5 vertex. Again recurse.
Assume neighbors are colored all differently.
Otherwise done.
Switch green to blue in component.
Done. Unless blue-green path to blue.
Switch orange to red in its component.
Done. Unless red-orange path to red.
Planar. ⟹ paths intersect at a vertex!
What color is it?
Must be blue or green to be on that path.
Must be red or orange to be on that path.
Contradiction. Can recolor one of the neighbors.
And recolor “center” vertex.
Four Color Theorem

**Theorem:** Any planar graph can be colored with four colors.

**Proof:** Not Today!

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Complete Graph.

$K_n$ complete graph on $n$ vertices.
All edges are present.
Everyone is my neighbor.
Each vertex is adjacent to every other vertex.

How many edges?
Each vertex is incident to $n - 1$ edges.
Sum of degrees is $n(n - 1)$.
$\implies$ Number of edges is $n(n - 1)/2$.
Remember sum of degree is $2|E|$.  

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$K_4$ and $K_5$

$K_5$ is not planar.
Cannot be drawn in the plane without an edge crossing!
Prove it! We did!

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A Tree, a tree.

Graph $G = (V,E)$.
Binary Tree!

More generally.

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Trees.

Definitions:
A connected graph without a cycle.
A connected graph with $|V| - 1$ edges.
A connected graph where any edge removal disconnects it.
A connected graph where any edge addition creates a cycle.

Some trees.

no cycle and connected? Yes.
$|V| - 1$ edges and connected? Yes.
removing any edge disconnects it. Harder to check. but yes.
Adding any edge creates cycle. Harder to check. but yes.
To tree or not to tree!

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Equivalence of Definitions.

**Theorem:**
"$G$ connected and has $|V| - 1$ edges" $\equiv$
"$G$ is connected and has no cycles."

**Lemma:** If $v$ is a degree 1 in connected graph $G$, $G - v$ is connected.

**Proof:**
For $x \neq v, y \neq v \in V$,
there is path between $x$ and $y$ in $G$ since connected.
and does not use $v$ (degree 1)
$\implies$ $G - v$ is connected.
Proof of only if.

\textbf{Thm:} “$G$ connected and has $|V| - 1$ edges” $\equiv$ “$G$ connected and has no cycles.”

\textbf{Proof of} $\implies$: By induction on $|V|$.

\textbf{Base Case:} $|V| = 1, 0 = |V| - 1$ edges and has no cycles.

\textbf{Induction Step:}

\textbf{Claim:} There is a degree 1 node.

\textbf{Proof:} First, connected $\implies$ every vertex degree $\geq 1$.

Sum of degrees is $2|V| - 2$

Average degree $2 - 2/|V|$

Not everyone is bigger than average!

By degree 1 removal lemma, $G - v$ is connected.

$G - v$ has $|V| - 1$ vertices and $|V| - 2$ edges so by induction $\implies$ no cycle in $G - v$.

And no cycle in $G$ since degree 1 cannot participate in cycle.

Recursive Definition.

A 0-dimensional hypercube is a node labelled with the empty string of bits.

An $n$-dimensional hypercube consists of a 0-subcube (1-subcube) which is an $n - 1$-dimensional hypercube with nodes labelled 0x (1x) with the additional edges (0x,1x).

Hypercubes.

Complete graphs, really connected! But lots of edges.

$|V|(|V| - 1)/2$

Trees, few edges. $|V|(|V| - 1)$

but just falls apart!

Hypercubes. Really connected. $|V|\log |V|$ edges!

Also represents bit-strings nicely.

$G = (V,E)$

$|V| = (0,1)^n$

$|E| = (x,y) |x$ and $y$ differ in one bit position.}

Proof of if

\textbf{Thm:} “$G$ is connected and has no cycles” $\implies$ “$G$ connected and has $|V| - 1$ edges”

\textbf{Proof:}

Walk from a vertex using untraversed edges.

Until get stuck.

\textbf{Proof of Claim:}

Can’t visit more than once since no cycle.

Entered. Didn’t leave. Only one incident edge.

Removing node doesn’t create cycle.

New graph is connected.

Removing degree 1 node doesn’t disconnect from Degree 1 lemma.

By induction $G - v$ has $|V| - 2$ edges.

$G$ has one more or $|V| - 1$ edges.

Tree’s fall apart.

\textbf{Thm:} There is one vertex whose removal disconnects $|V|/2$ nodes from each other.

Idea of proof.

Point edge toward bigger side.

Remove center node.

Hypercube: Can’t cut me!

\textbf{Thm:} Any subset $S$ of the hypercube where $|S| \leq |V|/2$ has $|S|$ edges connecting it to $V - S$; $|E \cap S \times (V - S)| \geq |S|$

Terminology:

$(S,V - S)$ is cut.

$(E \cap S \times (V - S))$ - cut edges.

Restatement: for any cut in the hypercube, the number of cut edges is at least the size of the small side.
**Proof of Large Cuts.**

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

**Proof:**

**Base Case:** \(n = 1\) \(V = \{0, 1\}\).

- \(S = \{0\}\) has one edge leaving. \(|S| = \phi\) has 0.

**Induction Step. Case 2.**

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

**Use recursive definition into two subcubes.**

- Two cubes connected by edges.
- Case 1: Count edges inside subcube inductively.
- Case 2: Count inside and across.

**Induction Step Idea**

- \((S_0, V - S_0)\) \(\in H_0\) \(\geq |S_0|\).
- \((S_1, V - S_1)\) \(\in H_1\) \(\geq |S_1|\).
- Total cut edges \(\geq |S_0| + |S_1| = |S|\).

**Induction Step**

- \((S, V - S)\) \(\in H\) \(\geq |S|\).
- \((S_0, V - S_0)\) \(\in H_0\) \(\geq |S_0|\).
- \((S_1, V - S_1)\) \(\in H_1\) \(\geq |S_1|\).
- Total cut edges \(\geq |S_0| + |S_1| = |S|\).

**Hypercubes and Boolean Functions.**

The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on \(\{0, 1\}^n\).

- Central area of study in computer science!
- Yes/No Computer Programs \(\equiv\) Boolean function on \(\{0, 1\}^n\)
- Central object of study.

**Have a nice weekend!**