CS 70 Discrete Mathematics and Probability Theory
Summer 2015 Chung-Wei Lin Final

PRINT Your Name: ______________________, ______________________
(last) (first)

SIGN Your Name: ______________________

PRINT Your Student ID: ______________________

CIRCLE Your Exam Room: 2050 VLSB 10 EVANS OTHER

Name of the person sitting to your left: ______________________

Name of the person sitting to your right: ______________________

The discussion section you usually attend:

×	#	Time	TA
	9	10–11am	Caroline
	2	11–12pm	Favian
	3	12–1pm	Nathan Z.
	10	1–2pm	Diana

×	#	Time	TA
	4	4–5pm	Jerry
	5	4–5pm	Kunal
	7	4–5pm	Nathan M.

×	#	Time	TA
	1	5–6pm	Lexing
	6	5–6pm	Akshay
	8	5–6pm	Anna

- After the exam starts, please write your student ID (or name) on every odd page (we will remove the staple when scanning your exam). You should not write anything, including your SID or name, after time is up.
- We will not grade anything on the extra pages (Pages 19–20) or outside of the space provided for a problem unless we are clearly told in the space provided for the question to look there.
- Except special notices, you must show your work to get credits.
- You may consult three single-sided sheets (or one double-sided sheet plus one single-sided sheet) of notes. Apart from that, you may not look at books, notes, etc. Calculators, phones, and computers are not permitted.
- You have 170 minutes. There are 17 questions for a total of 160 points plus 20 bonus points. Use the number of points as a rough guide for the amount of time to allocate to that question.
- There are 20 pages (sides) on the exam. Notify a proctor immediately if a page is missing.

Do not turn this page until your instructor tells you to do so.

Section 1: Straightforward Questions (68 points)

1. Counting and Probability (21 points, 3 points for each part)

For the following parts, just write down answers. You do not need to calculate the exact value of $\binom{a}{b}$, $a!$, or a^b. No explanation is required. No partial credit will be given.

(a) How many different anagrams of DISCRETE are there?

(b) There are 2015 students. How many different ways to choose 70 students and let them form a team?

(c) How many different subsets of $\{1, 2, 3, \ldots, 300\}$ are there?

(d) How many different non-negative integer solutions to $x_0 + x_1 + x_2 + x_3 + \cdots + x_{70} = 2015$? (Note that it is 0-indexed.)

(e) The birthdays of three students are all in March, 1997 (there are 31 days in March). Assume their birthdays are randomly and uniformly distributed. What is the probability that their birthdays are all different?

(f) There are 11 poker cards $\{\spadesuit$ A, $\spadesuit$ J, $\spadesuit$ Q, $\spadesuit$ K, $\heartsuit$ A, $\heartsuit$ 2, $\heartsuit$ 3, $\diamondsuit$ A, $\clubsuit$ A, $\clubsuit$ 4, $\clubsuit$ 5$\}$. You draw a card and it is an Ace (A); what is the probability that the card is the Heart ($\heartsuit$) suit?

(g) The number of typos per board is modelled by the Poisson distribution:

$$\Pr[X = i] = \frac{\lambda^i}{i!} e^{-\lambda} \qquad \text{for } i = 0, 1, 2, \ldots$$

If there is 1 typo per board on average, what is the probability that there are 2 typos on a board?

2. Distribution, Expectation, and Variance (9 points, 3 points for each part)

For the following parts, just write down answers. No explanation is required. No partial credit will be given.

(a) A special dice has 2 sides with "0", 3 sides with "1", and 1 side with "3". Each side has the same probability of being rolled. Let random variable X be the outcome of the dice. What is the distribution of X?

(b) What is the expectation of X? Calculate the exact value.

(c) What is the variance of X? Calculate the exact value.

3. Markov's Inequality and Chebyshev's Inequality (9 points, 3 points for each part)

For the following parts, just write down answers. No explanation is required. No partial credit will be given.

(a) Given a positive discrete random variable X with its expectation 10, use Markov's inequality to provide an upper bound on $\Pr[X \geq 30]$?

(b) Following Part (a), the variance of X is 5, use Chebyshev's inequality to provide an upper bound on $\Pr[|X - 10| \geq 20]$?

(c) Following Parts (a) and (b), provide an upper bound (this bound should be smaller than the answer in Part (a)) on $\Pr[X \geq 30]$?

4. Stable Marriage Algorithm (3 points)

Find a female optimal stable pairing for the following case. No explanation is required. No partial credit will be given.

Men	Preference Lists
1	C > A > B
2	B > C > A
3	B > A > C

Women	Preference Lists
A	1 > 2 > 3
B	2 > 1 > 3
C	2 > 3 > 1

5. Modular Arithmetic and RSA (6 points, 3 points for each part)

For the following parts, just write down answers. No explanation is required. No partial credit will be given.

(a) What is the multiplicative inverse of 6 (mod 7)?

(b) Assume $p = 3, q = 11, e = 7, d = 3$ in the RSA cryptography; if the receiver receives 4, what is the original message?

6. Polynomials and Error Correction Codes (6 points, 3 points for each part)

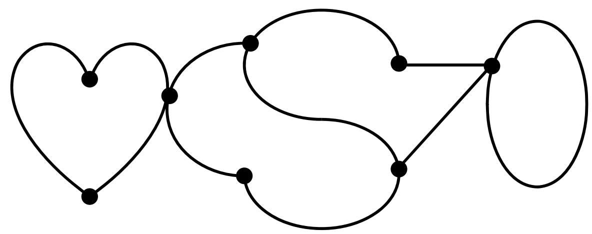
For the following parts, just write down answers. No explanation is required. No partial credit will be given.

(a) Working over $GF(7)$, there is a polynomial $P(x)$ with degree at most 2. Given $P(0) = 3$, $P(1) = 0$, and $P(2) = 0$, what is the value of $P(3)$? Calculate the exact value.

(b) Kevin wants to send a message of 10 ordered packets to Stuart. No matter how many packets are sent, there are at most 5 lost packets (erasure errors) and 2 corrupted packets (general errors). What is the minimum number of packets Kevin should send so that Stuart can recover the message?

7. TRUE or FALSE — First Round (14 points, 1 point for each part)

For any discrete random variable X, Y, Z, any real number C, and any proposition P, Q, determine whether the following statements are true or false. Just circle the correct choice. No explanation is required. No partial credit will be given.



(a) *T* *F* There exists an Eulerian *walk* for the graph above.

(b) *T* *F* $\mathbb{E}(X+C) = \mathbb{E}(X) + C$.

(c) *T* *F* $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$.

(d) *T* *F* $\text{Var}(X+C) = \text{Var}(X)$.

(e) *T* *F* $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$.

(f) *T* *F* If you flip a fair coin $2n$ times, the probability that you get exactly n Heads increases as n increases.

(g) *T* *F* The set $\{1, 2, 3\}$ is countably infinite.

(h) *T* *F* $\mathbb{N}$, the set of all natural numbers, is countably infinite.

(i) *T* *F* $\mathbb{Z}$, the set of all integers, is countably infinite.

(j) *T* *F* $\mathbb{Q}$, the set of all rational numbers, is countably infinite.

(k) *T* *F* The set of all (n, q) where $n \in \mathbb{N}$ and $q \in \mathbb{Q}$, is countably infinite.

(l) *T* *F* $\mathbb{R}$, the set of all real numbers, is countably infinite.

(m) *T* *F* $(P \Longrightarrow Q) \Longrightarrow (Q \Longrightarrow P)$.

(n) *T* *F* $\forall y \exists x\, P \Longrightarrow \exists x \forall y\, P$.

Section 2: 8-Point Questions (16 points)

8. Multinomial Distribution (8 points)

There are 3 coins:

- Coin 1 has Head probability $\frac{1}{3}$.
- Coin 2 has Head probability $\frac{1}{2}$.
- Coin 3 has Head probability $\frac{3}{4}$.

You repeat the following random experiment 100 times:

- You flip Coin 1 first. If you get Head, you flip Coin 2; otherwise, you flip Coin 3.

What is the expectation of the number of Heads? Note that you flip coins 200 times in total. Calculate the exact value.

9. Geometric Distribution (8 points)

Coco is trying to collect four coupons, C_1, C_2, C_3, C_4. She does not know which coupon(s) she will get before paying. There are two stores selling coupons:

- In Store A, she pays 3 dollars to get a coupon, and the probability of getting each coupon is always $\frac{1}{4}$.
- In Store B, C_1 and C_2 are always packaged together, and C_3 and C_4 are also always packaged together. She pays 8 dollars to get one package (with two coupons), and the probability of getting each pair (either (C_1, C_2) or (C_3, C_4)) is always $\frac{1}{2}$.

Which store should she go, or it does not matter?

Section 3: 10-Point Questions (40 points)

10. Variance — Second Round (10 points)

Two friends have n books $(n > 0)$ they would both like to read. Each friend (independently of the other) picks a random permutation of the books and reads them in that order, one book per week (for n consecutive weeks). Let X be the number of weeks in which both of them are reading the same book. What is the variance of X? (Your answer should *not* include summation $\sum$ or the following form: $a_1 + a_2 + a_3 + \cdots + a_n$.)

11. Chebyshev's Inequality — Second Round (10 points)

You are estimating the Head probability p of a coin. You flip the coin n times and estimate p as $\widehat{p} = \frac{1}{n}S_n$, where S_n is the number of Heads in the n tosses. Use Chebyshev's inequality to find n such that $\Pr\left[|\widehat{p} - p| \geq 0.1\sqrt{p}\right] \leq 0.2$.

12. Complete Graphs (10 Points)

There are n cities in the Minions Land. There is exactly one highway between each pair of cities (no self-loop), and each highway is bidirectional (a minion can traverse from City A to City B or from City B to City A through the highway between them). Kevin wants to start from Berkeley (one of the cities) and *visit every other city exactly once before returning to Berkeley* (he cannot return to Berkeley and then visit another city). When he is at a city, he decides the next step by randomly and uniformly choosing one *untraversed* highway. What is the probability that he can visit every other city exactly once before returning to Berkeley?

13. Hashing (10 Points)

A hash table has 99 locations, and m keys are mapped independently but not uniformly. The probability that a key is mapped to Location 1 is $\frac{2}{100}$, and the probability that a key is mapped to each of other 98 locations is $\frac{1}{100}$. How many keys can be hashed so that the probability of a collision is less than $\frac{1}{2}$? Try to maximize m but guarantee the requirement is satisfied. (Hint: consider an event that two keys have a collision and then consider the union of those events.)

Section 4: 12-Point Questions (36 points)

14. Load Balancing (12 Points)

There are n identical jobs and $2n$ identical processors. Jobs are assigned to the processors independently and uniformly. Let A be the event that the load of some processor is at least 3 jobs. What is n so that $\Pr[A] \leq \frac{1}{2}$? Try to maximize n but guarantee the requirement is satisfied. (Hint: consider an event that the load of a specific processor is at least 3 jobs and the union of those events.)

15. TRUE or FALSE — Last Round (12 points)

For the following statement, claim TRUE or FALSE first. If you claim TRUE, prove it. If you claim FALSE, disprove it.

Statement: for all integers a, b where $a \geq b > 0$ and $\gcd(a, b) = 1$, $\gcd(a+b, a-b) = 1$ or 2.

TRUE FALSE

16. Trees (12 points)

Given an integer $n \geq 1$ and an undirected graph as follows:

$$V = \{v_0, v_1, v_2, \ldots, v_{3n}\},$$

and

$$E = \{\{v_0, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{3n-1}, v_{3n}\}\} \cup \{\{v_{3n}, v_1\}, \{v_0, v_{n+1}\}, \{v_0, v_{2n+1}\}\}.$$

If you want to get a tree (still connecting all vertices) by removing some edges in E, how many different trees can you get? Write down your answer as a polynomial of n, *i.e.*, $a_d n^d + a_{d-1} n^{d-1} + \cdots + a_1 n + a_0$.

Section 5: Bonus Question (20 points)

17. Super Combo (20 points)

There are n vertices $\{v_1, v_2, \ldots, v_n\}$ on a plane, and the coordinates of v_i are (x_i, y_i). The bounding box of v_i and v_j is a rectangular defined by $\{(x,y) \mid (x-x_i)(x-x_j) \leq 0 \text{ and } (y-y_i)(y-y_j) \leq 0\}$. An edge is added between two vertices if and only if there is no other vertex inside or on the boundary of the bounding box of the two vertices. Assume the coordinates of each vertex are randomly and uniformly decided. Provide an upper bound on the expected number of edges (we are not looking for a trivial upper bound n^2). You can get more points for providing better (*i.e.*, smaller) upper bounds.

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