CS 70 Discrete Mathematics and Probability Theory
Summer 2015 Chung-Wei Lin Discussion 1C

1. Correct Proofs?

Given $a, b, c, d \in \mathbb{R}$, Is there any flaw in the following proofs? If there is a flaw, fix the proof if the claim is true or provide a counterexample if the claim is false.

(a) **Claim**: if $a < c$ and $b < d$, then $a + b < c + d$.
Proof:

$$\begin{aligned} a < c &\implies a + b < c + b \quad \text{(add } b \text{ to both sides)} \\ &\implies a + b < c + d \quad \text{(because } b < d\text{)} \end{aligned}$$

Answer: This is a correct proof.

(b) **Claim**: if $a + b < c + d$, then $a < c$ and $b < d$.
Proof:

$$\begin{aligned} a < c &\implies a + b < c + b \quad \text{(add } b \text{ to both sides)} \\ &\implies a + b < c + d \quad \text{(because } b < d\text{)} \end{aligned}$$

Answer: The proof should start from what is given true ($a + b < c + d$ in this proof), or, if it starts from what to be proved, every step should be an if-and-only-if derivation (the last step is not in this proof). The claim is not true, and a counterexample is $(a, b, c, d) = (1, 1, 5, 0)$.

(c) **Claim**: if $a < b$, then $2a + 3 < 2b + 3$.
Proof:

$$\begin{aligned} a < b &\implies 2a < 2b \quad \text{(multiply 2 to both sides)} \\ &\implies 2a + 3 < 2b + 3 \quad \text{(add 3 to both sides)} \end{aligned}$$

Answer: This is a correct proof.

(d) **Claim**: if $2a + 3 < 2b + 3$, then $a < b$.
Proof:

$$\begin{aligned} a < b &\implies 2a < 2b \quad \text{(multiply 2 to both sides)} \\ &\implies 2a + 3 < 2b + 3 \quad \text{(add 3 to both sides)} \end{aligned}$$

Answer: The proof should start from what is given true ($2a + 3 < 2b + 3$ in this proof), or, if it starts from what to be proved, every step should be an if-and-only-if derivation (it only shows one-direction derivations in this proof). The claim is true, and here is a fixed proof:

$$\begin{aligned} 2a + 3 < 2b + 3 &\implies 2a < 2b \quad \text{(subtract 3 from both sides)} \\ &\implies a < b \quad \text{(multiply } 1/2 \text{ to both sides)} \end{aligned}$$

Here is another fixed proof:

$$
\begin{aligned}
a < b &\iff 2a < 2b \quad \text{(multiply 2 to both sides)} \\
&\iff 2a+3 < 2b+3 \quad \text{(add 3 to both sides)}
\end{aligned}
$$

2. Proofs

(a) A *perfect square* is an integer n of the form $n = m^2$ for some integer m. Prove that every odd perfect square is of the form $8k+1$ for some integer k.

(b) Can you use another approach to prove it?

Answer:

(a) Let $n = m^2$ for some integer m. Since n is odd, m is also odd, so it is of the form $m = 4\ell+1$ or $m = 4\ell+3$ for some integer ℓ. We consider both cases:

- If $m = 4\ell+1$ for some integer ℓ, then $m^2 = 16\ell^2+8\ell+1 = 8(2\ell^2+\ell)+1$.
- If $m = 4\ell+3$ for some integer ℓ, then $m^2 = 16\ell^2+24\ell+9 = 8(2\ell^2+3\ell+1)+1$.

In both cases, we have $n = m^2 = 8k+1$ for some integer k.

(b) Let $n = m^2$ for some integer m. Since n is odd, m is also odd, *i.e.*, of the form $m = 2\ell+1$ for some integer ℓ. Then, $m^2 = 4\ell^2+4\ell+1 = 4\ell(\ell+1)+1$. Since one of ℓ and $\ell+1$ must be even, $\ell(\ell+1)$ is of the form $2k$ and $n = m^2 = 8k+1$.

3. Proof by Contraposition

Prove that if $a+b < c+d$, then $a < c$ or $b < d$.

Answer: Assume $a \geq c$ and $b \geq d$ (note that this is the negation of $a < c \vee b < d$). Then, $a+b \geq c+b \geq c+d$, which is the negation of $a+b < c+d$.

4. Proof by Contradiction

Prove that if you put $n+1$ apples into n boxes, any way you like, then at least one box must contain at least 2 apples. This is known as the *pigeonhole principle*.

Answer: Suppose this is not the case. Then, all the boxes would contain at most 1 apple, and thus the maximum number of apples we could have would be n. This is a contradiction since we have $n+1$ apples.