The Traditional Propose & Reject Algorithm

- **Every Morning**: Each man proposes to the most preferred woman on his list who has not yet rejected him.
- **Every Afternoon**: Each woman collects all the proposals she received in the morning; to the man she likes best, she responds “maybe, come back tomorrow” (she now has him on a string), and to the others, she says “never”.
- **Every Evening**: Each rejected man crosses off the woman who rejected him from his list.

The above loop is repeated each successive day until each woman has a man on a string; on this day, each woman marries the man she has on a string.

1. Stable Marriage

    Consider the following preference lists for 4 men (1, 2, 3 and 4) and 4 women (A, B, C and D):

    | Men | Preferences | Women | Preferences |
    |-----|-------------|-------|-------------|
    | 1   | A > D > C > B | A     | 1 > 3 > 4 > 2 |
    | 2   | B > D > A > C | B     | 3 > 1 > 4 > 2 |
    | 3   | D > A > B > C | C     | 2 > 1 > 3 > 4 |
    | 4   | D > A > B > C | D     | 4 > 3 > 2 > 1 |

    (a) Run the Traditional Propose & Reject Algorithm with the preference lists above and list the final stable couples created.

    (b) Now, assume that 1 and A are at the top of all the preference lists for the women and men respectively. What can you say about them in every stable pairing?

    (c) Now, assume that 4 and D are at the bottom of all the preference lists for the women and men respectively. What can you say about them in every stable pairing?
2. Universal Preferences

Suppose that preferences are universal, that is all $n$ men share the preferences $W_1 > W_2 > \ldots > W_n$ and all women share the preferences $M_1 > M_2 > \ldots > M_n$.

(a) What result do we get from running the algorithm with men proposing?
(b) What result do we get from running the algorithm with women proposing?
(c) What does this tell us about the number of stable matchings?

3. Proofs about the Traditional Propose & Reject Algorithm

Prove the following statements about the Traditional Propose & Reject Algorithm:

(a) In any execution of the algorithm, if a woman receives a proposal in day $i$, she receives some proposal on every day thereafter until termination.
(b) In any execution of the algorithm that takes $k$ days, there must be some woman who does not receive a proposal in day $k - 1$.
(c) In any execution of the algorithm, if woman $W$ receives no proposal in day $i$, then she receives no proposal in any previous day $j$, $1 \leq j < i$.
(d) From the above parts, can you prove that in any execution of the algorithm, there is at least one woman who only receives a single proposal?