CS 70 Discrete Mathematics and Probability Theory
Summer 2015 Chung-Wei Lin Discussion 3D

1. Pokemon Counting!

(a) I have caught 30 different Pokemon[1] so far. In how many ways can I choose a team of 6, such that the order of my team matters?

$30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25$. (30 choices for the first slot, 29 for the second slot, and so on.)

(b) For this part and the next two, you can assume we no longer care about the order of the Pokemon. In how many ways can I choose a team of 6 under this assumption?

$\binom{30}{6}$

(c) Among my 30 caught Pokemon, only 4 can learn the move "Fly". In how many ways can I choose a team of 6, such that there is **exactly** one Pokemon who knows the move "Fly"?

$\binom{4}{1} \cdot \binom{26}{5}$

(d) In how many ways can I choose a team of 6, such that there is **at least** one Pokemon who knows the move "Fly"?

This is the same as the number of ways to choose six Pokemon subtract the number of ways to choose six Pokemon, where none of which knows the move "Fly". $\binom{30}{6} - \binom{26}{6}$. We can also enumerate the possibilities $(1,2,3,4$ Pokemon that know how to fly), but it will take longer.

(e) You were victorious against the Elite Four, and Professor Oak generously invited all six members of your team to the Hall of Fame. Suppose he wants to sit your team in a circular table for dinner (which consists of only Oran Berries), in how many ways can he do so?

$\frac{6!}{6} = 5!$. Why are we dividing by 6? It's because if each Pokemon move one seat to the right (or to the left), it's the same seating! And there are 6 different ways the Pokemon can move and still preserve the same ordering. Alternatively, we can think of this as ordering 5 Pokemon in a straight line, then bend the line to form an almost-circle, and the last Pokemon only has one position to fit in to form a circle.

(f) Suppose Charizard and Pikachu, two members of your team, want to sit next to each other. How would your answer to the above question change?

$2 \cdot \frac{5!}{5} = 2 \cdot 4! = 48$. Treat Charizard and Pikachu as one Pokemon (Charichu), and then carry out the same step above under the assumption that there's only 5 Pokemon to be seated. Note that the two Pokemon can swap seats in $2! = 2$ ways, so we multiply by 2.

(g) Suppose Meowth and Pikachu, two members of your team, don't want to sit next to each other. In how many ways can Professor Oak arrange the seatings?

$5! - 2 \cdot \frac{5!}{5} = 5! - 2 \cdot 4! = 72$.

2. Pokemon Anagrams

An anagram of a word is any re-ordering of the letters of the word, in any order. It does not have to be an English word or an actual Pokemon name.

[1] Pokemon is an abbreviation of Pocket Monsters. Hence, the plural of Pokemon is still Pokemon...

(a) How many different anagrams of PIKACHU are there?

Since the order of the letters matter and all of the letters are distinct, there are 7!

(b) How many different anagrams of KADABRA are there?

If we first pretend that the 3 A's are all distinct, then there are 7! anagrams. But since the 3 A's are identical, we counted each anagram an extra 3! ways. Hence, there are $7!/3!$ anagrams total. Another way to think about this: we first choose 3 of out of the 7 possible positions to place the A, there are $\binom{7}{3}$ choices. There are then 4 positions left to place the K, 3 positions to place the D, 2 positions to place the B, and one position to place the R, so there are $\binom{7}{3}(4!) = 7!/3!$ anagrams total.

(c) How many different anagrams of RATTATA are there?

Since the T's and A's are identical and appear three times each, the answer is $\frac{7!}{3!3!}$.

3. Pokemon Levels

For this question, assume that a Pokemon's *level* is a **nonnegative integer**, unless stated otherwise[2].

(a) Suppose you need to train your team (technical word: "level grinding") until the total levels of all six Pokemon is exactly 20 to qualify for your first Gym battle. Let x_i denotes the level of the ith Pokemon. How many nonnegative integer solutions are there to this equation?

$$x_0 + x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

$\binom{25}{5}$. Draw out 20 stars (20 levels), then place 5 bars among the stars to separate the stars into 6 bars. If two separators are next to each other, it means the Pokemon's level is 0, or 0 star goes into that bin. There are 25 objects, or positions, and we need 5 of them to place a bar, hence $\binom{25}{5}$.

(b) In general, how many solutions does

$$x_0 + x_1 + \ldots + x_k = n$$

have, if all x_i must be non-negative integers?

$\binom{n+k}{k}$. There is a bijection between a sequence of n ones and k plusses and a solution to the equation: x_0 is the number of ones before the first plus, x_1 is the number of ones between the first and second plus, etc. A key idea is that if a bijection exists between two sets they must be the same size, so counting the elements of one tells us how many the other has.

(c) Suppose I only need to "grind" two of my Pokemon for a Double battle challenge. How many solutions does

$$x_0 + x_1 = n$$

have, if every x_i must be *strictly positive* integers?

$n-1$. It's easy just to enumerate the solutions here. x_0 can take values $1, 2, \ldots, n-1$ and this uniquely fixes the value of x_1. So, we have $n-1$ ways to do this. This is just an example of the more general question below.

(d) What if I need to train all of my n Pokemon, as seen in Part (b)? How many solutions does

$$x_0 + x_1 + \ldots + x_k = n$$

have, if every x_i must be *strictly positive* integers?

$\binom{(n-(k+1))+k}{k} = \binom{n-1}{k}$. By subtracting 1 from all $k+1$ variables (i.e., $k+1$ from the total required), we reduce it to problem with the same form as the previous problem. Once we have a solution to that we reverse the process, and adding 1 to all the non-negative variables gives us positive variables.

[2] This is actually not true in the game. See the last part.

(e) Most of the constraints above are actually incorrect in the game – a Pokemon cannot have level0, and most Pokemon caught in the wild or bred from eggs actually start at level 5. Using your knowledge from the above parts, solve for the number of solutions of

$$x_0 + x_1 + x_2 + x_3 + x_4 + x_5 = 100,$$

where $x_i \geq 5$.

$\binom{75}{5}$. Similar to the above part, we subtract 5 from each variable, so a total of 30. We then reduce the problem to the same one in Part (b).