1. Variance

Prove or give a counterexample: for any random variables $X$ and $Y$, $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$.

Let $X$ be any random variable such that $\text{Var}(X) \neq 0$. Then, if we set $Y = X$, $\text{Var}(X + Y) = \text{Var}(2X) = E((2X)^2) - (E(2X))^2 = 4(E(X^2) - E(X)^2) = 4\text{Var}(X) \neq \text{Var}(X) + \text{Var}(Y)$.

2. Cookies

TAs are giving out cookies to students in section! To make it more interesting, we give exactly 10 cookies to a student with probability $\frac{1}{2}$, otherwise we give either 1, 2, or 3 cookies randomly with equal probability. What is the expected number of cookies you’d get? What is its variance?

$$E(X) = \frac{1}{2} \cdot 10 + \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 = 5 + \frac{1}{6} + \frac{2}{6} + \frac{3}{6} = 6.$$ \[ \text{Var}(X) = E(X^2) - (E(X))^2 = \left( \frac{1}{2} \cdot 10^2 + \frac{1}{6} \cdot 1^2 + \frac{1}{6} \cdot 2^2 + \frac{1}{6} \cdot 3^2 \right) - (6)^2 = \frac{157}{3} - 36 = \frac{49}{3}. \]

3. Bernoulli and Binomial Distribution

A random variable $X$ is called a Bernoulli random variable with parameter $p$ if $X = 1$ with probability $p$ and $X = 0$ with probability $1 - p$.

(a) Calculate $E(X)$ and $\text{Var}(X)$.

$$E(X) = p \cdot 1 + (1 - p) \cdot 0 = p; \quad \text{Var}(X) = E((X - E(X))^2) = E(X^2) - E(X)^2 = p \cdot 1^2 + (1 - p) \cdot 0^2 - p^2 = p(1 - p).$$

(b) A Binomial random variable with parameters $n$ and $p$ is defined to be the sum of $n$ independent, identically distributed Bernoulli random variables with parameter $p$. If $Z$ is a Binomial random variable with parameters $n$ and $p$, what are $E(Z)$ and $\text{Var}(Z)$?

Let $X_i$ be i.i.d. Bernoulli random variables with parameter $p$. By linearity of expectation, $E(Z) = \sum_{i=1}^{n} E(X_i) = np$. Moreover, since $X_i$’s are independent, $\text{Var}(Z) = \sum_{i=1}^{n} \text{Var}(X_i) = np(1 - p)$. 
4. Will I Get My Package?

A deceitful delivery dude is out transporting $n$ packages to $n$ customers. Not only does he hand a random package to each customer, but he also opens a package before delivering with probability $\frac{1}{2}$ (independently of the choice of the package). Let $X$ be the number of customers who receive their own packages unopened.

(a) Compute the expectation $E(X)$.

Define $X_i = \begin{cases} 1 & \text{if the } i\text{-th customer gets his/her own package unopened;} \\ 0 & \text{otherwise.} \end{cases}$

$E(X) = E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i)$.

$E(X_i) = \Pr[X_i = 1] = \frac{1}{2n}$ since the $i$-th customer will get his/her package with probability $\frac{1}{n}$ and it is unopened with probability $\frac{1}{2}$ and packages are opened independently.

Hence $E(X) = n \cdot \frac{1}{2n} = \frac{1}{2}$. By linearity: $E(X) = E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i) = n \cdot \frac{1}{2n} = \frac{1}{2}$.

(b) Calculate the probability that two particular customers $i, j$ receive their own packages unopened.

$\Pr[\text{both customers receive own packages}] = \frac{(n-2)!}{n!}$.

$\Pr[i = \text{unopened}, j = \text{unopened}] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

Since the two events are independent, $\Pr[X_i = 1, X_j = 1] = \frac{1}{4n(n-1)}$.

(c) Compute $\text{Var}(X)$.

$\text{Var}(X) = E(X^2) - E(X)^2$.

$E(X^2) = E((X_1 + X_2 + \ldots + X_n)^2) = E(\sum_{i,j} X_i X_j) = \sum_{i,j} E(X_i X_j)$, where the last equality follows from using linearity of expectation.

If $i = j$ then $E(X_i X_j) = E(X_i^2) = \frac{1}{2n}$; if $i \neq j$, then $E(X_i X_j) = \Pr[X_i X_j = 1] \cdot 1 + \Pr[X_i X_j = 0] \cdot 0 = \frac{1}{4n(n-1)}$.

Hence, $E(X^2) = \sum_{i,j} E(X_i X_j) = \sum_i E(X_i^2) + \sum_{i \neq j} E(X_i X_j) = n \cdot \frac{1}{2n} + n(n-1) \cdot \frac{1}{4n(n-1)} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$.

Thus, $\text{Var}(X) = E(X^2) - E(X)^2 = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$. 

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