CS 70 Discrete Mathematics and Probability Theory
Summer 2015 Chung-Wei Lin Discussion 7D

1. Counting Edge

Consider an undirected graph G with n nodes, no multi-edges, and self-loops allowed.

(a) What is the maximum number of edges G can have?

(b) How many different G graphs can there be? Graphs with the same number of nodes are only different if they have different sets of edges (assume nodes are labeled).

(c) Consider a directed graph H with n nodes, multi-edges not allowed (edges (a,b), and (b,a) are considered different), and self-loops not allowed. Edge (c,d) denotes that node c knows node d (similarly, edge (d,c) denotes node d knows node c). What is the maximum number of edges H can have?

(d) Assume that any given edge exists randomly and independently with probability p. Pick any node u. What is the probability all other nodes know u, but u does not know any other nodes?

(e) If the same node u from the previous part does not know any other nodes, but all other nodes know u, can such a graph have a Eulerian tour?

2. Number of Heads Revisited

Consider an undirected graph G with n nodes, no multi-edges, and no self-loops. Consider all possible edges between any two nodes in G, and let e be the number of total possible edges. Suppose the probability, p, that a given edge exists between any two nodes is independent. (Answer in terms of e).

(a) What is the probability that exactly 34 edges exist in this graph? (Assume $e \geq 34$).

(b) What is the expected number of edges the graph will have? What is the variance?

(c) Consider an undirected graph with no self-loops, no multi-edges, and with four nodes a,b,c,d, and probability that an edge exists between any two nodes is still p. What is the probability that the graph is connected?

3. Introduction to Trees

Recall that a tree is a connected graph with no cycles, (and so no self-loops, and no multi-edges). Show that any tree with at least 2 nodes must have a node of degree 1.

4. Graph Gardening

Prove that if graph G is a tree with e edges and n nodes, then $e = n - 1$. Use induction on n.