CS 70 Discrete Mathematics and Probability Theory
Summer 2015 Chung-Wei Lin Discussion 8B

1. Countability

Consider degree-one polynomials, i.e. polynomials of the form $P(x) = ax + b$. Determine if the set of all degree-one polynomials is countable under the following conditions:

(a) a and b must be integers.
Countable, there is a bijection to $\mathbb{Z} \times \mathbb{Z}$, which we showed in lecture was countable.

(b) a and b must be rational numbers.
Countable, there is a bijection to $\mathbb{Q} \times \mathbb{Q}$. We showed $\mathbb{Q}$ was countable, so this is also countable.

(c) a and b are real numbers.
Uncountable, there is a bijection with $\mathbb{R} \times \mathbb{R}$. Since $\mathbb{R}$ is uncountable, then this is also uncountable.

2. Injection, Surjection, or Bijection?

For each of the following functions from $\mathbb{R}$ to $\mathbb{R}$, determine whether it is an injection, surjection, bijection, or none of the above.

(a) $f(x) = 2^x$
Injection. $f(x)$ cannot take on non-positive values.

(b) $f(x) = x^2$
None. Not an injection since every non-zero $f(x)$ occurs twice. Not a surjection because $f(x)$ cannot take on negative values.

(c) $f(x) = 2x + 1$
Injection, Surjection, and Bijection. There is exactly one x that maps to any given value, namely $f^{-1}(y) = (y-1)/2$.

3. Union of Countable Sets

Prove that if A is countable and B is countable, then $A \cup B$ is countable.

Proof: Direct proof. Since A is countable, there exists a bijection from A to a subset of $\mathbb{N}$. Since B is countable, there exists a bijection from B to a subset of $\mathbb{N}$. Consider the bijection from $\mathbb{N}$ to nonnegative even numbers. Using that bijection and the bijection from A to $\mathbb{N}$, there exists a bijection from A to a subset of nonnegative even numbers. Using the bijection from $\mathbb{N}$ to positive odd numbers and the bijection from B to $\mathbb{N}$, there exists a bijection from B to a subset of positive odd numbers. This means that $A \cup B$ has a bijection onto a subset of the union of nonnegative even numbers and positive odd numbers, which is just $\mathbb{N}$. This means that $A \cup B$ is countable.

4. Another Proof

Prove that if A is uncountable and B is a countable subset of A, then $A - B$ is uncountable.

We will use proof by contradiction. Suppose for contradiction that A is uncountable, B is a countable subset of A, and $A - B$ is countable. This means that there exists a bijection from $A - B$ to a subset of $\mathbb{N}$. Since B is

countable, this means that there exists a bijection from B to a subset of $\mathbb{N}$. Consider a bijection from $A - B$ to (possibly a subset of) nonnegative even numbers and a bijection from B to (possibly a subset of) positive odd numbers. This means that "combining" the two bijections gives us a bijection from A to (possibly a subset of) $\mathbb{N}$. However, if there exists a bijection from A to a subset of $\mathbb{N}$, then A is countable. This is a contradiction on the assumption that A is uncountable, and thus if A is uncountable and B is a countable subset of A, then $A - B$ is uncountable.