1. **Countability**

Consider degree-one polynomials, i.e. polynomials of the form \( P(x) = ax + b \). Determine if the set of all degree-one polynomials is countable under the following conditions:

(a) \( a \) and \( b \) must be integers.

Countable, there is a bijection to \( \mathbb{Z} \times \mathbb{Z} \), which we showed in lecture was countable.

(b) \( a \) and \( b \) must be rational numbers.

Countable, there is a bijection to \( \mathbb{Q} \times \mathbb{Q} \). We showed \( \mathbb{Q} \) was countable, so this is also countable.

(c) \( a \) and \( b \) are real numbers.

Uncountable, there is a bijection with \( \mathbb{R} \times \mathbb{R} \). Since \( \mathbb{R} \) is uncountable, then this is also uncountable.

2. **Injection, Surjection, or Bijection?**

For each of the following functions from \( \mathbb{R} \) to \( \mathbb{R} \), determine whether it is an injection, surjection, bijection, or none of the above.

(a) \( f(x) = 2^x \)

Injection. \( f(x) \) cannot take on non-positive values.

(b) \( f(x) = x^2 \)

None. Not an injection since every non-zero \( f(x) \) occurs twice. Not a surjection because \( f(x) \) cannot take on negative values.

(c) \( f(x) = 2x + 1 \)

Injection, Surjection, and Bijection. There is exactly one \( x \) that maps to any given value, namely \( f^{-1}(y) = (y - 1)/2 \).

3. **Union of Countable Sets**

Prove that if \( A \) is countable and \( B \) is countable, then \( A \cup B \) is countable.

Proof: Direct proof. Since \( A \) is countable, there exists a bijection from \( A \) to a subset of \( \mathbb{N} \). Since \( B \) is countable, there exists a bijection from \( B \) to a subset of \( \mathbb{N} \). Consider the bijection from \( \mathbb{N} \) to nonnegative even numbers. Using that bijection and the bijection from \( A \) to \( \mathbb{N} \), there exists a bijection from \( A \) to a subset of nonnegative even numbers. Using the bijection from \( \mathbb{N} \) to positive odd numbers and the bijection from \( B \) to \( \mathbb{N} \), there exists a bijection from \( B \) to a subset of positive odd numbers. This means that \( A \cup B \) has a bijection onto a subset of the union of nonnegative even numbers and positive odd numbers, which is just \( \mathbb{N} \). This means that \( A \cup B \) is countable.

4. **Another Proof**

Prove that if \( A \) is uncountable and \( B \) is a countable subset of \( A \), then \( A - B \) is uncountable.

We will use proof by contradiction. Suppose for contradiction that \( A \) is uncountable, \( B \) is a countable subset of \( A \), and \( A - B \) is countable. This means that there exists a bijection from \( A - B \) to a subset of \( \mathbb{N} \). Since \( B \) is
countable, this means that there exists a bijection from $B$ to a subset of $\mathbb{N}$. Consider a bijection from $A - B$ to (possibly a subset of) nonnegative even numbers and a bijection from $B$ to (possibly a subset of) positive odd numbers. This means that “combining” the two bijections gives us a bijection from $A$ to (possibly a subset of) $\mathbb{N}$. However, if there exists a bijection from $A$ to a subset of $\mathbb{N}$, then $A$ is countable. This is a contradiction on the assumption that $A$ is uncountable, and thus if $A$ is uncountable and $B$ is a countable subset of $A$, then $A - B$ is uncountable.