if it is voltage-controlled, and by

$$v = v(q) \tag{1-30}$$

if it is charge-controlled. For a voltage-controlled capacitor, the current entering the capacitor can be expressed in a form analogous to Eq. (1-28); thus

$$i(t) = \frac{dq(t)}{dt} = \frac{dq(v)}{dv}\frac{dv(t)}{dt}$$

or

$$i(t) = C(v(t))\frac{dv(t)}{dt} \tag{1-31}$$

where

$$C(v) \equiv \frac{dq(v)}{dv} \tag{1-32}$$

is called the *incremental capacitance* of the capacitor. Notice that the incremental capacitance is a function of the capacitor voltage and becomes a constant only in the case of a linear capacitor.

Exercise 1: A typical nonlinear capacitor is characterized by the v-q curve $q = kv^{3/2}$, where k is a physical parameter. (*a*) Find the incremental capacitance $C(v)$. (*b*) If the applied voltage is given by $v(t) = \frac{1}{4}\cos^2 t$, find the charge $q(t)$ and the current $i(t) \equiv dq(t)/dt$. (*c*) Calculate $i(t)$ by using Eq. (1-31).

Exercise 2: An abrupt-junction diode is a semiconductor p-n junction which behaves like a capacitor, provided the voltage across the junction is less than 0.5 volt. Its incremental capacitance is given by $C(v) = k(\phi - v)^{-1/n}$, where k, ϕ, and n are constants which depend upon the parameters of the device. (*a*) Plot the incremental capacitance on logarithmic paper for the range $-100 < v < 0.5$ volt. (Assume $k = 80 \times 10^{-12}$, $\phi = 0.5$, and $n = 2$.) (*b*) What are the maximum and the minimum values of the capacitance (in picofarads or 10^{-12} F) within this range of applied voltage? (*c*) Do you have sufficient information to recover the v-q curve? If not, what additional information do you need?

1-7-3 SOME PRACTICAL APPLICATIONS OF TWO-TERMINAL NONLINEAR CAPACITORS

What are nonlinear capacitors good for? Can they do useful things which nonlinear resistors cannot? The answer to the second question is obviously yes, for otherwise we would not be studying them. In addition to being able to do a number of things described earlier for resistors, a nonlinear capacitor can do better in certain cases. Although we do not yet have the background necessary to demonstrate this assertion, suffice it to say that both nonlinear resistors and capacitors are capable of generating higher harmonics. However, with an appropriate design, it is possible to extract more output power in any given harmonic component from a nonlinear capacitor than from a nonlinear resistor. This means that a nonlinear-capacitor-frequency multiplier has a higher efficiency than a nonlinear-resistor-frequency multiplier. In addition to this application, a few of the many other useful functions are briefly described as follows.

Frequency division In many practical systems, it is desirable to convert a given sinusoidal signal of frequency ω_1 into another sinusoidal signal of a lower frequency ω_2; namely, $\omega_2 = \omega_1/n$, where n is an integer. In this case, the lower-frequency output signal is said to be a *subharmonic* of the higher-frequency output signal. It can be shown that *a nonlinear resistor cannot generate subharmonics.* To demonstrate that a nonlinear capacitor can generate a subharmonic signal, consider a nonlinear capacitor whose incremental capacitance is given by

$$C(v) = \left[\frac{1 - \sqrt{1 - v^2}}{2(1 - v^2)}\right]^{1/2} \tag{1-33}$$

If we apply a voltage $v(t) = \sin \omega t$ across this capacitor, the current $i(t)$ can be calculated from Eqs. (1-31) and (1-33); thus

$$\begin{aligned} i(t) &= \left[\frac{1 - \sqrt{1 - \sin^2 \omega t}}{2(1 - \sin^2 \omega t)}\right]^{1/2} (\omega \cos \omega t) \\ &= \left(\frac{1 - \cos \omega t}{2 \cos^2 \omega t}\right)^{1/2} (\omega \cos \omega t) \\ &= \omega \sqrt{\frac{1 - \cos \omega t}{2}} = \omega \sin \frac{\omega}{2} t \end{aligned} \tag{1-34}$$

Hence, the output current is a sinusoid with frequency equal to half the original frequency. The phenomenon of subharmonic generation by a nonlinear capacitor has been utilized in many practical applications. One application consists of utilizing the two "distinct" frequencies as the two distinct states in designing a digital computer. Another interesting application consists of converting the high-frequency output of a laser beam into a lower-frequency signal.

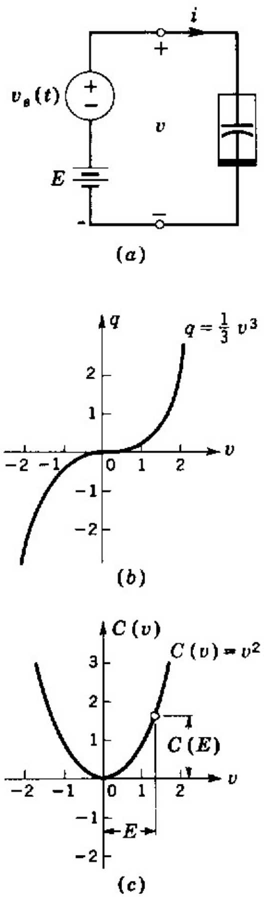


Fig. 1-18. A nonlinear capacitor can be used as a tuning element by varying the dc voltage E across the capacitor.

Parametric amplifier Just as with nonlinear resistors, it is possible to generate beat-frequency components by applying two sinusoidal signals of frequencies ω_1 and ω_2 in series with a nonlinear capacitor. It can be shown that if one of the two signals (say $v_1 = A \sin \omega_1 t$) is very weak while the other signal (say $v_2 = B \sin \omega_2 t$) is very strong, it is possible to extract (with the help of filters) the signal with frequency ω_1 and at the same time greatly amplify its amplitude, say $v_o = 1{,}000\, A \sin \omega_1 t$. The result is that we have an amplifier. For reasons that we are not equipped to elaborate here, this amplifier is called a *parametric amplifier.* It is widely used in artificial satellites because it has some definite advantages over conventional amplifiers.

Electronic tuning Suppose we connect a voltage source $v_s(t)$ and a battery with terminal voltage E in series with a nonlinear capacitor as shown in Fig. 1-18*a*. For simplicity, let the v-q curve be given by $q = \frac{1}{3} v^3$ as shown in Fig. 1-18*b*. Then its incremental capacitance is given by $C(v) = v^2$, as shown in Fig. 1-18*c*. Now in many electronic systems, such as a radio receiver, the signal $v_s(t)$ is very small (say, a few millivolts) compared with the value of the dc voltage E. Hence, for most practical purposes, the incremental capacitance

$$C(v) = C(v_s(t) + E) \approx C(E) \tag{1-35}$$

can be considered to depend *only* on the value of E. In this case, Eq. (1-31) becomes

$$i(t) = C(E)\frac{dv}{dt} \tag{1-36}$$

Since $C(E)$ is no longer a function of time, Eq. (1-36) is identical with Eq. (1-28) which describes a linear capacitor. The only difference is that we can change the value of the capacitance by simply changing the value of E. This observation is of great practical importance. One immediate application is in the area of *electronic tuning.* The conventional way to tune a radio receiver from one station to another is to turn a knob which moves the tuning dial. Any one who opens up the cover of a radio receiver would recognize that this tuning knob is used to rotate the plates of an air capacitor, thereby changing the value of its capacitance. In other words, the standard tuning process consists of adjusting the value of a capacitor *mechanically.* This operation can now be replaced by a nonlinear capacitor connected as shown in Fig. 1-18*a* where the tuning is accomplished by adjusting the voltage *E*. This method is clearly far superior to the use of bulky air capacitors. In fact, this technique of electronic tuning is fast becoming a standard method in electronic systems.

Exercise 1: Find the incremental capacitance $C(v)$ required to generate a 30-Hz subharmonic sinusoidal current waveform from an input voltage $v(t) = 100 \cos 120\pi t$. HINT: Make use of the trigonometric identity

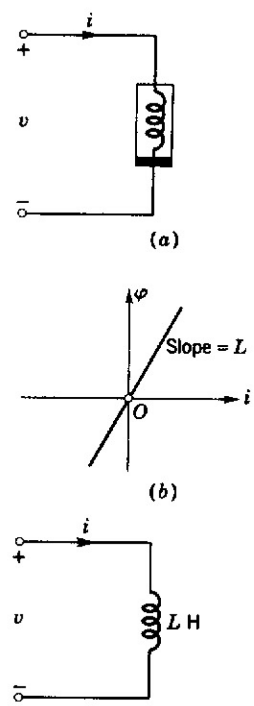
$$\cos\frac{x}{2} = \sqrt{\frac{1+\cos x}{2}}$$

Exercise 2: A common nonlinear capacitor used for electronic tuning is the *varactor diode*. It is characterized by a v-q curve $q(v) = -(3/2)C_0\phi_0(1 - v/\phi_0)^{2/3}$, where C_0 and ϕ_0 are constants which vary from device to device. When $v = 0$, the incremental capacitance was measured to be equal to 60 pF. (*a*) Derive the incremental capacitance $C(v)$. (*b*) If $\phi_0 = 0.35$, find the range of the input voltage required to tune the capacitance from 5 to 100 pF. To operate the varactor as a nonlinear capacitor, the voltage must not exceed 0.35 volt.

1-8 TWO-TERMINAL INDUCTORS

A two-terminal black box which can be characterized by a curve in the i-φ plane is called a *two-terminal inductor* and will be denoted by the symbol shown in Fig. 1-19*a*. The darkened edge of this symbol has the same significance as before.

Fig. 1-19. Symbols for a two-terminal inductor.



1-8-1 LINEAR INDUCTORS

An important subclass of inductors can be characterized by a straight line through the origin of the i-φ plane as shown in Fig. 1-19*b*. This subclass is called *linear inductors* and will be denoted by the conventional symbol shown in Fig. 1-19*c*. A linear inductor can be described analytically by

$$\varphi = Li \tag{1-37}$$

where the constant L represents the slope of the straight line and is called the *inductance* associated with the inductor. The unit of inductance is the *henry* and will be denoted by H. To find the voltage across a linear inductor, we substitute Eq. (1-37) for φ in Eq. (1-8) and obtain

$$v(t) = L\frac{di(t)}{dt} \tag{1-38}$$

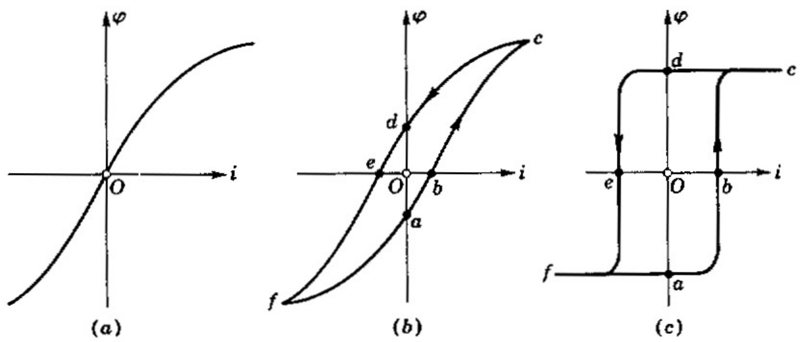


Fig. 1-20. The i-φ curve of three practical nonlinear inductors.

A linear inductor is therefore completely characterized by one number, namely, its inductance. Again, we would differentiate between the terms *inductor* and *inductance.*

1-8-2 NONLINEAR INDUCTORS

If an inductor is characterized by an i-φ curve other than a straight line through the origin, it is called a *nonlinear inductor.* In this case, the inductor can no longer be described by a single number, and hence the entire i-φ curve must be given. For example, Fig. 1-20*a* shows the i-φ curve of a typical nonlinear inductor.

Another common nonlinear inductor consists of a coil wound around an iron core. Its i-φ curve (obtained by applying a sinusoidal current excitation) is shown in Fig. 1-20*b*. This curve is a multivalued function of both i and φ and is commonly referred to as the *hysteresis loop.*[1] Observe that starting at point a with $i = 0$, the flux linkage φ increases with i along the path a-b-c. Upon reaching point c when φ attains its maximum value, the flux linkage φ does not retrace the original path. Instead, it decreases with the current i along the path c-d-e-f. Upon reaching point f when i attains its minimum value, the flux linkage φ returns to point a to complete the loop. The shape of the hysteresis loop depends on the type of material used for the core. For certain materials, the hysteresis loop is almost rectangular, as shown in Fig. 1-20*c*.

We shall denote the i-φ curve of a nonlinear inductor by

$$\varphi = \varphi(i) \tag{1-39}$$

if it is current-controlled, and by

$$i = i(\varphi) \tag{1-40}$$

[1] Actually, this hysteresis loop is a valid description only under the assumption that the current waveform is sinusoidal. For other periodic excitations, the hysteresis loop becomes much more complicated. A *complete* characterization of elements described by hysteresis loops is a very difficult and still unsolved problem.

if it is flux-controlled. In the case of a current-controlled inductor, the voltage across the inductor can be expressed in a form analogous to Eq. (1-38); thus

$$v(t) = \frac{d\varphi(t)}{dt} = \frac{d\varphi(i)}{di}\frac{di(t)}{dt}$$

or

$$v(t) = L(i(t))\frac{di(t)}{dt} \tag{1-41}$$

where

$$L(i) \equiv \frac{d\varphi(i)}{di} \tag{1-42}$$

is called the *incremental inductance* of the inductor. Notice that for a linear inductor, the incremental inductance coincides with the inductance, as it should.

Exercise 1: The i-φ curve of a certain nonlinear inductor can be represented approximately by the cubic equation $\varphi = i^3$. If the inductor is connected across a current source with terminal current $i_s(t) = \sin t$, find and sketch the incremental inductance $L(i)$ and the inductor voltage $v(t)$.

Exercise 2: An inductor is said to be the "dual" of a capacitor, and vice versa, because there exists a one-to-one correspondence between the two elements. Exhibit a list of corresponding quantities.

1-8-3 SOME PRACTICAL APPLICATIONS OF TWO-TERMINAL NONLINEAR INDUCTORS

What are nonlinear inductors good for? Where are they used in practice? To answer these questions would again require more background than we have at present. However, it is instructive to describe a few simple applications.

Frequency conversion Just as is true of capacitors, a nonlinear inductor is capable of generating both *harmonics* and *subharmonics* of a given sinusoidal signal. It can be shown to have the same efficiency as does a nonlinear capacitor. This property is widely used in telephone systems.

Memory and storage Consider the rectangular hysteresis curve shown in Fig. 1-20*c*. Observe that when $i = 0$, φ may assume either one of two distinct values (point *a* or point *d*) depending on the previous history of the excitation current. These two distinct states can be used to represent the two states (0 and 1) in a digital computer. When many of these elements are combined properly, the result is a "memory" or "storage" device to store present information for future use. While there are many other candidates, this memory device has some significant advantages. One is that in both states $i = 0$, and hence no power is being consumed. Since hundreds and thousands of these elements are used in a practical computer, the saving in power cost is enormous.

1-9 ENERGY AND POWER

The energy flow into a two-terminal black box during any time interval (t_0,t_1) is by definition the time integral of power from t_0 to t_1; namely,

$$w(t_0,t_1) = \int_{t_0}^{t_1} v(t)i(t)\,dt \tag{1-43}$$

Since $w(t_0,t_1)$ is a *relative* quantity depending on the time interval (t_0,t_1), it is convenient for us to define another related but *absolute* quantity by letting t_0 equal zero and t_1 approach infinity, and then take the average of the energy flow over the entire infinite time interval; namely,

$$P_{av} \equiv \lim_{t_1\to\infty} \frac{w(0,t_1)}{t_1} \tag{1-44}$$

Since the quantity P_{av} has the dimension of energy per second, it is called the average power. Substituting Eq. (1-43) into Eq. (1-44), we obtain the explicit expression

$$P_{av} = \lim_{t_1\to\infty} \frac{1}{t_1} \int_0^{t_1} v(t)i(t)\,dt \tag{1-45}$$

To illustrate the use of this formula, let us calculate the average power entering a 4-Ω linear resistor due to an applied voltage $v(t) = 2 \sin \pi t$; thus

$$P_{av} = \lim_{t_1\to\infty} \frac{1}{t_1} \int_0^{t_1} (2 \sin \pi t)\left(\frac{2 \sin \pi t}{4}\right) dt \tag{1-46}$$

$$P_{av} = \lim_{t_1\to\infty} \left(½ - \frac{\sin 2\pi t_1}{4\pi t_1}\right) = ½$$

In the case where the voltage $v(t)$ and current $i(t)$ are periodic functions with commensurate periods T_v and T_i, respectively, the power $p(t) = v(t)i(t)$ will also be periodic. However, the period of $p(t)$ is not necessarily equal to T_v or T_i. For the example considered in Eq. (1-46), $T_v = T_i = 2$, but the period of $p(t)$ is 1. If we denote the *minimum* period of $p(t)$ by T, then

$$p(t + nT) = v(t + nT)i(t + nT) = p(t) \tag{1-47}$$

In this case, it is more convenient to let $t_1 = nT$ and rewrite Eq. (1-45) in the equivalent form:

$$\begin{aligned}
P_{\text{av}} &= \lim_{n\to\infty} \frac{1}{nT} \int_0^{nT} v(t)i(t)\,dt \\
&= \lim_{n\to\infty} \frac{1}{nT}\left[\int_0^T v(t)i(t)\,dt + \int_T^{2T} v(t)i(t)\,dt \right.\\
&\qquad\qquad\qquad\qquad \left. + \cdots + \int_{(n-1)T}^{nT} v(t)i(t)\,dt\right] \\
&= \lim_{n\to\infty} \frac{1}{nT}\left[n \int_0^T v(t)i(t)\,dt\right] \\
&= \lim_{n\to\infty} \frac{1}{T} \int_0^T v(t)i(t)\,dt
\end{aligned}$$

Since the variable n no longer appears in this integral, the limit operation is superfluous and can be removed. Hence, *for periodic signals*, the average power can be written in the following simplified but equivalent form:

$$P_{\text{av}} = \frac{1}{T} \int_0^T v(t)i(t)\,dt = \frac{w(0,T)}{T} \tag{1-48}$$

where T is the minimum period of $v(t)i(t)$. Applying this formula to the same example considered in Eq. (1-46), we obtain

$$P_{\text{av}} = \frac{1}{1} \int_0^1 (2 \sin \pi t)\left(\frac{2 \sin \pi t}{4}\right) dt = \frac{1}{2}$$

as we should.

Exercise: The voltage and current waveforms of a two-terminal black box are given, respectively, by $v = \sin (3.14)t$ and $i = \sin \pi t$. (*a*) Show that even though both $v(t)$ and $i(t)$ are periodic, the power $p(t)$ is not periodic. (*b*) For most practical purposes, $p(t)$ is said to be "almost periodic." Explain why.

The three expressions given by Eqs. (1-43), (1-45), and (1-48) are valid for any two-terminal black box. Let us now consider the special cases where the black box consists of a single nonlinear resistor, capacitor, or inductor. In so doing, we shall be able to derive a number of useful relationships. We shall also be able to draw some very important physical interpretations. Let us consider the three cases one at a time.

Case 1: Two-terminal nonlinear resistor Consider the nonlinear resistor shown in Fig. 1-21*a* and the three common types of *v*-*i* curves shown in Fig. 1-21*b*, *c*, and *d*. The *v*-*i* curve can be described in the functional form by $i = i(v)$ if it is voltage-controlled, or by $v = v(i)$ if it is current-controlled. A strictly monotonically increasing *v*-*i* curve can obviously be described by either $i = i(v)$ or $v = v(i)$. Accordingly, the instantaneous power flow $p_R(t)$, energy flow $w_R(t_0,t_1)$, and average power $P_{R_{av}}$ can be determined and are tabulated in Table 1-2 for these three cases.

Observe that corresponding to any operating point Q at any time t, the instantaneous power $p_R(t)$ is simply equal to the area of the shaded rectangles shown in Fig. 1-21. This power must, of course, come from the energy supplied by the external circuit connected across the resistor. From Table 1-2 we observe that the expressions for $p_R(t)$, $w_R(t_0,t_1)$, and $P_{R_{av}}$ depend on two pieces of information, namely,

1. The *v*-*i* curve
2. The voltage waveform $v(t)$ or the current waveform $i(t)$

Hence, in order to find out what happens to the power that enters the resistor, we must be given these two pieces of information. For

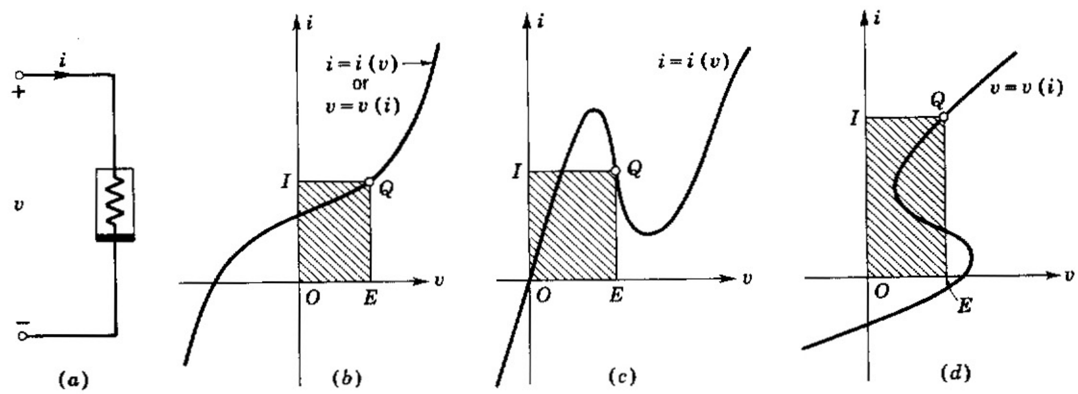


Fig. 1-21. The instantaneous power absorbed by a nonlinear resistor at any time t_0 is equal numerically to the area of the rectangle formed by the v,i axes and a vertex Q with coordinates $(v(t_0),i(t_0))$.

TABLE 1-2 *Instantaneous power, energy, and average power flow in a nonlinear resistor.*

	Strictly monotonically increasing v-i curve	Voltage-controlled v-i curve	Current-controlled v-i curve
$p_R(t)$	$= v(t)i(v(t))$ $= i(t)v(i(t))$	$= v(t)i(v(t))$	$= i(t)v(i(t))$
$w_R(t_0,t_1)$	$= \int_{t_0}^{t_1} v(t)i(v(t))\,dt$ $= \int_{t_0}^{t_1} i(t)v(i(t))\,dt$	$= \int_{t_0}^{t_1} v(t)i(v(t))\,dt$	$= \int_{t_0}^{t_1} i(t)v(i(t))\,dt$
$p_{R_{av}}$	$= \lim_{t_1\to\infty} \frac{1}{t_1}\int_0^{t_1} v(t)i(v(t))\,dt$ $= \lim_{t_1\to\infty} \frac{1}{t_1}\int_{t_0}^{t_1} i(t)v(i(t))\,dt$	$= \lim_{t_1\to\infty} \frac{1}{t_1}\int_0^{t_1} v(t)i(v(t))\,dt$	$= \lim_{t_1\to\infty} \frac{1}{t_1}\int_0^{t_1} i(t)v(i(t))\,dt$

example, suppose the v-i curve is represented by $i = v^3$, and the voltage is given by $v(t) = 2 \sin \pi t$. The instantaneous power can then be calculated; thus

$$p_R(t) = (2 \sin \pi t)(2 \sin \pi t)^3 = 16(\sin \pi t)^4$$

Observe that $p_R(t)$ has a period $T = 1$. The energy flow during the time interval $(0,t_1)$ and the average power due to the periodic signal are given, respectively, by

$$w_R(0,t_1) = 6t_1 - \frac{3}{\pi}\sin 2\pi t_1 - \frac{4}{\pi}(\sin \pi t_1)^3(\cos \pi t_1)$$

and

$$p_{R_{av}} = \frac{w_R(0,T)}{T} = \frac{w_R(0,1)}{1} = 6 \tag{1-49}$$

Equation (1-49) shows that even though the voltage $v(t)$ changes from positive to negative values periodically, there is a net positive average power flow entering the resistor. Since this power is not returned to the external circuit whenever the voltage returns to its initial value during each period, it cannot be recovered and is therefore said to be "lost" or "dissipated" in the resistor. Since energy cannot be destroyed, this loss of electrical energy in the resistor is merely transformed into heat energy.

The average power for the above example is positive. Let us now consider another example where this is not true. Suppose the