CHAPTER

ONE

KIRCHHOFF'S LAWS

As an electrical engineer, one needs to analyze and design circuits. Electric circuits are present almost everywhere, in home computers, television and hi-fi sets, electric power networks, transcontinental telecommunication systems, etc. Circuits in these applications vary a great deal in nature and in the ways they are analyzed and designed. The purpose of this book is to give an introductory treatment of circuit theory which covers considerable breadth and depth. This differs from a traditional introductory course on circuits, which is restricted to "linear" circuits and covers mainly circuits containing the classical *RLC* elements.

The first chapter deals with the fundamental postulates of lumped-circuit theory, namely, Kirchhoff's laws. Naturally, we need to explain the word "lumped" first. It is also important to understand the concept of "modeling." For example, in circuit theory we first model a "physical circuit" made of electric devices by a "circuit" which is an interconnection of circuit elements. Since Kirchhoff's laws hold for any lumped circuit, the discussion can be dissociated with the electrical properties of circuit elements, which will be treated in the succeeding chapters.

A key concept introduced in this chapter is the representation of a circuit by a graph. This allows us to deal with multiterminal devices in the same way as we would with a conventional two-terminal device. In addition, it enables us to give a formal treatment of Kirchhoff's laws and a related fundamental theorem, Tellegen's theorem.

1 THE DISCIPLINE OF CIRCUIT THEORY

Circuit theory is the fundamental engineering discipline that pervades all electrical engineering. For the present, by *physical circuit* we mean any *interconnection* of (physical) *electric devices*. Familiar examples of electric devices are resistors, coils, condensers, diodes, transistors, operational amplifiers (op amps), batteries, transformers, electric motors, electric generators, etc.

The goal of circuit theory is to *predict* the electrical *behavior* of physical circuits. The purpose of these predictions is to improve their design: in particular, to decrease their cost and improve their performance under all conditions of operation (e.g., temperature effects, aging effects, possible fault conditions, etc.).

Circuit theory is an *engineering discipline* whose domain of application is extremely broad. For example, the *size* of the circuits varies enormously: from large-scale integrated circuits which include hundreds of thousands of components and which fit on a fingernail to circuits found in radios, TV sets, electronic instruments, small and large computers, and finally, to telecommunications circuits and power networks that span continents. The *voltages* encountered in the study of circuits vary from the microvolt (μV) [e.g., in noise studies of precision instruments—to megavolts (MV) of power networks]. The *currents* vary from femtoamperes ($1\text{ fA} = 10^{-15}$ A) [e.g., in electrometers—to megaamperes (MA)] encountered in studies of power networks under fault conditions. The *frequencies* encountered in circuit theory vary from zero frequency [direct current (dc) conditions] to tens of gigahertz ($1\text{ GHz} = 10^9$ Hz) encountered in microwave circuits. The *power levels* vary greatly from 10^{-14} watts (W) for the incoming signal to a sensitive receiver (e.g., faint radio signals from distant galaxies) to electric generators producing $10^9\text{ W} = 1000$ megawatts (MW).

Circuit theory focuses on the *electrical* behavior of circuits. For example, it does not concern itself with thermal, mechanical, or chemical effects. Its aim is to predict and explain the (terminal) *voltages* and (terminal) *currents* measured at the device *terminals*. It does not concern itself with the physical phenomena occurring inside the device (e.g., in a transistor or in a motor). These considerations are covered in device physics courses and in electrical machinery courses.

The goal of circuit theory is to make quantitative and qualitative predictions on the electrical behavior of circuits; consequently the tools of circuit theory will be mathematical, and the *concepts* and *results* pertaining to circuits will be expressed in terms of circuit equations and *circuit variables*, each with an obvious operational interpretation.

2 LUMPED-CIRCUIT APPROXIMATION

Throughout this book we shall consider only *lumped circuits*. For a physical circuit to be considered *lumped*, its physical dimension must be small enough so that, for the problem at hand, electromagnetic waves propagate across the circuit virtually instantaneously. Consider the following two examples:

Example 1 Consider a small computer circuit on a chip whose extent is, say, 1 millimeter (mm); let the shortest signal time of interest be 0.1 nanosecond [$\frac{1}{10}$ of a nanosecond (ns) $= 10^{-10}$ of a second (s)]. Electromagnetic waves travel at the velocity of light, i.e., 3×10^8 meters per second (m/s); to travel 1 mm, the time elapsed is 10^{-3} m/(3×10^8 m/s) $= 3.3 \times 10^{-12}$ s $= 0.0033$ ns. Therefore the propagation time in comparison with the shortest signal time of interest is negligible. More generally, let d be the largest dimension of the circuit, Δt the shortest time of interest, and c the velocity of light. If $d \ll c \cdot \Delta t$, then the circuit may be considered to be lumped.

Example 2 Consider an audio circuit: The highest frequency of interest is, say, $f = 25$ kHz. For electromagnetic waves, this corresponds to a wavelength of $\lambda = c/f = (3 \times 10^8 \text{ m/s})/(2.5 \times 10^4 \text{ s}^{-1}) = 1.2 \times 10^4$ m $=$ 12 km $\cong$ 7.5 miles. So even if the circuit is spread across a football stadium, the size of the circuit is very small compared to the shortest wavelength of interest λ. More generally, if $d \ll \lambda$, the circuit may be considered to be lumped.

When these conditions are satisfied, electromagnetic theory proves[1] and experiments show that the lumped-circuit approximation holds; namely, throughout the physical circuit the current $i(t)$ through any device terminal and the voltage difference $v(t)$ across any pair of terminals, at any time t, are well-defined. A circuit that satisfies these conditions is called a *lumped circuit*.

From an electromagnetic theory point of view, a lumped circuit reduces to a point since it is based on the approximation that electromagnetic waves propagate through the circuit instantaneously. For this reason, in lumped-circuit theory, the respective locations of the elements of the circuit will not affect the behavior of the circuit. The approximation of a *physical* circuit by a *lumped* circuit is analogous to the modeling of a rigid body as a particle: In doing so, all the data relating to the extent (shape, size, orientation, etc.) of the body are ignored by the theory.

Thus, lumped-circuit theory is related to the more general electromagnetic theory by an approximation (propagation effects are neglected). This is analogous to the relation of classical mechanics to the more exact relativistic mechanics: Classical mechanics delivers excellent predictions provided the velocities are much smaller than the velocity of light. Similarly, when the above conditions hold, lumped-circuit theory delivers excellent predictions of physical circuit behavior.

In situations where lumped approximation is not valid, the physical dimensions of the circuit must be considered. To distinguish such circuits from

[1] R. M. Fano, L. J. Chu, and R. M. Adler, *Electromagnetic Fields, Energy and Forces*, John Wiley and Sons, New York, 1960.

lumped circuits we call them *distributed circuits*. Typical examples of distributed circuits are circuits made of waveguides and transmission lines. In distributed circuits the current and voltage variables would depend not only on time, but also on space variables such as length and width. We need electromagnetic theory for predictions of the behavior of distributed circuits and for analysis and design. In this book we restrict our treatment to lumped circuits.

3 ELECTRIC CIRCUITS, MODELS, AND CIRCUIT ELEMENTS

By *electric device* we mean the physical object in the laboratory or in the factory, for example, the coil, the capacitor, the battery, the diode, the transistor, the motor, etc. *Physical circuits* are obtained by connecting electric devices by wires. Most of the time, these wires will be assumed to be perfectly conducting. We think of these electric devices in terms of *idealized models* like the resistor ($v = Ri$), the inductor ($v = L\, di/dt$), the capacitor ($i = C\, dv/dt$), etc., that you have studied in physics.

Note that these *idealized models* are precisely defined; to distinguish them from electric devices we call them *circuit elements*. It is important to distinguish between a coil made of a fine wire wrapped around a ferrite torus—an *electric device*—and its model as an inductor, or as a resistor in series with an inductor—a *circuit element*, or a combination of circuit elements.

Every model is an approximation. Depending on the application or the problem under consideration, the same physical device may be approximated by several different models. Each of these models is an interconnection of (idealized) circuit elements. For example, we will encounter several different models for the operational amplifier (op amp).[2]

Any *interconnection* of *circuit elements* is called a *circuit*. Thus a circuit is an interconnection of (idealized) models of the corresponding physical devices. The relation between physical circuits and circuits is illustrated in Fig. 3.1. If the (theoretical) predictions based on *analysis* of the circuit do not agree with the measurements, the cause of the disagreement may lie at any step of the process (e.g., erroneous measurement, faulty analysis, etc.). One frequent cause is a poor choice of model, e.g., using a low-frequency model outside of its frequency range of validity, or a linear model outside its amplitude range of validity.

Our subject is circuit theory, consequently we consider the models of the electric devices constituting the physical circuit as given at the outset; our goal is to develop methods to predict the behavior of the circuit. Note that we say "circuit," not "physical circuit": Past experience, however, does give us the

[2] Analogously, in classical mechanics a communications satellite circling the earth may be modeled as a particle, or a rigid body, or an elastic body depending on the problem being studied.

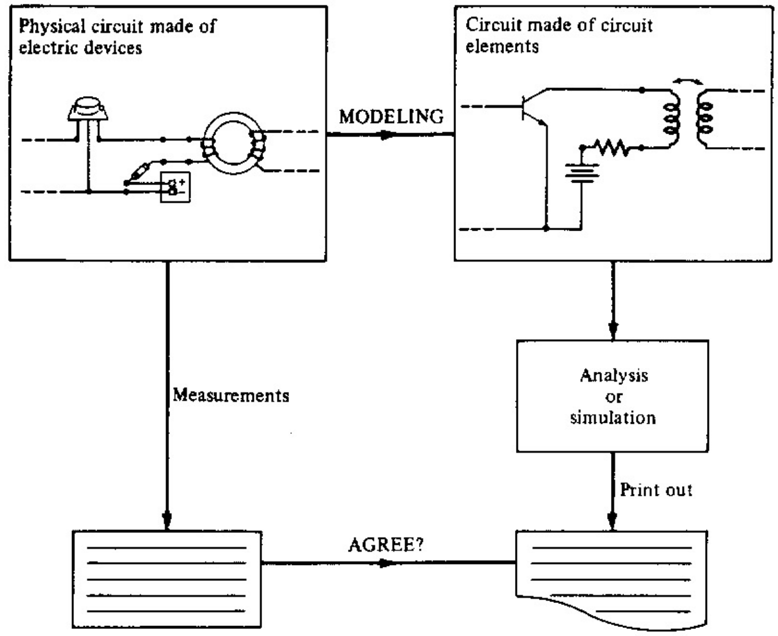

Figure 3.1 Illustration of the relation between physical circuits and circuits, between physical devices and circuit elements, and between laboratory measurements and circuit analysis.

confidence that given any physical circuit we can model it by a circuit which will adequately predict its behavior.

In Fig. 3.2*a* we show a *physical circuit* made up of *electric devices*: a generator, resistor, transistor, battery, transformer, and load. To analyze the physical circuit, we first model it with the circuit shown in Fig. 3.2*b*, which is an interconnection of *circuit elements*: voltage sources, resistors, a capacitor,

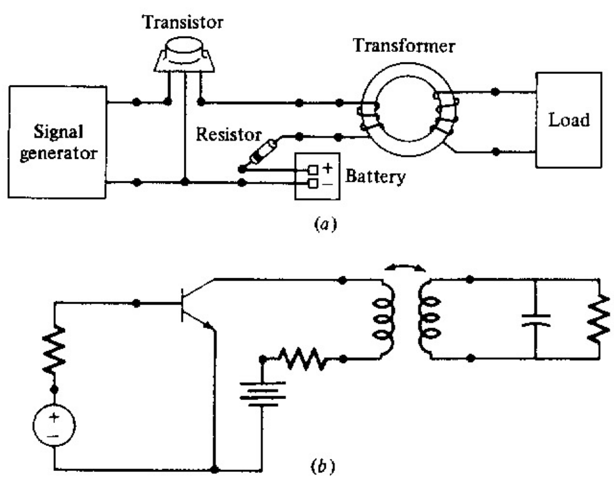

Figure 3.2 (*a*) Physical circuit made of electric devices and (*b*) its circuit model made of circuit elements.

coupled inductors, and a transistor represented by their usual symbols. The electrical properties of some of the two-terminal elements (voltage sources and resistors) will be discussed in Chap. 2, and that of the multiterminal elements (transistor and ideal transformer) will be treated in Chap. 3.

When electric devices are interconnected, we use conducting wires to tie the terminals together as shown in Fig. 3.2*a*. When circuit elements are interconnected, we delete the conducting wires and merge the terminals to obtain the circuit in Fig. 3.2*b*. A *node* is any junction in a circuit where terminals are joined together or any isolated terminal of a circuit element, which is not connected. The circuit in Fig. 3.2*b* has eight nodes (marked with heavy dots). With the introduction of the concept of a node, we are ready to formally treat the subject of interconnection and state the two fundamental postulates of circuit theory, namely, Kirchhoff's voltage law and Kirchhoff's current law.

4 KIRCHHOFF'S LAWS

In lumped circuits, the voltage between any two nodes and the current flowing into any element through a node are well-defined.[3] Since the *actual* direction of current flow and the *actual* polarity of voltage difference in a circuit can *vary* from one instant to another, it is generally impossible to specify in advance the *actual* current direction and voltage polarity in a given circuit. Just as in classical mechanics where it is essential to set up a "frame of reference" from which the actual instantaneous positions of a system of particles can be uniquely specified, so too must we set up an "electrical frame of reference" in a circuit in order that currents and voltages may be *unambiguously* measured.

4.1 Reference Directions

To set up an electrical reference frame, we assign *arbitrarily* a *reference direction* to each current variable by an arrow, and a *reference polarity* to each voltage variable by a pair of plus (+) and minus (−) signs, as illustrated in Fig. 4.1 for two-terminal, three-terminal, and n-terminal elements.[4]

On each terminal lead we indicate an arrow called the *current reference direction*. It plays a crucial role. Consider Fig. 4.1*a*. If at some time t_0, $i_2(t_0) = 2$ A, it means that, at time t_0, a current of 2 A flows *out* of the two-terminal element of Fig. 4.1*a* by node ②. If, at some later time t_1,

[3] We assume that the circuit is connected; the definition of "connectedness" will be given later.

[4] An example of a six-terminal element is the filter at the output of an audio amplifier: It directs the high frequencies to the tweeter and the low frequencies to the woofer. (Later we shall see that such a filter may also be viewed as a three-port.)

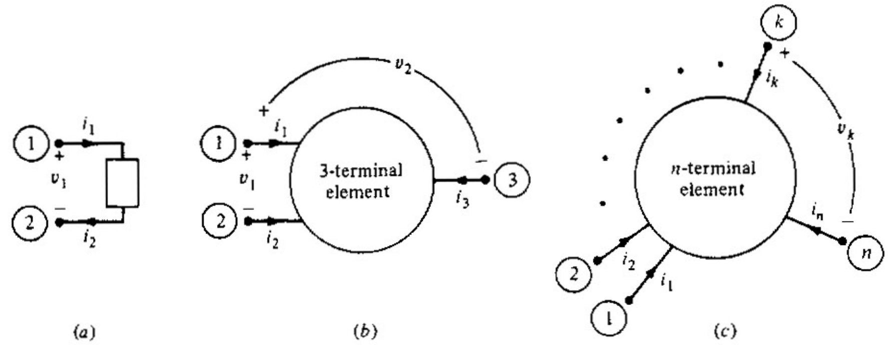

Figure 4.1 Illustration of reference directions using two-, three-, and n-terminal elements.

$i_2(t_1) = -25$ mA, it means that, at time t_1, a current of 25 mA flows *into* the two-terminal element by node ②.

The point is that the current reference direction together with the sign of $i(t)$ determines the actual direction of the flow of electric charges.

On Fig. 4.1 we assign + and − signs to pairs of terminals, e.g., in Fig. 4.1*b* the pair ①, ② and the pair ①, ③. These signs indicate the *voltage reference direction*. Consider Fig. 4.1*a*. If, at some time t_0, $v_1(t_0) = 3$ millivolts (mV), it means that, at time t_0, the electric potential of terminal ① is 3 mV *larger* than the electric potential of terminal ②. Similarly, considering Fig. 4.1*c*, if at time t_1, $v_k(t_1) = -320$ V, it means that the electric potential of terminal ⓚ is, at time t_1, 320 V *smaller* than the electric potential of terminal ⓝ.

Exercise Write down the physical meaning of the following statements in Fig. 4.1*c*: $i_k(t_1) = -2$ mA, $i_2(t_1) = 4$ A, $-v_k(t_1) = 5$ V.

4.2 Kirchhoff's Voltage Law (KVL)

Given any *connected* lumped circuit having n nodes, we may choose (arbitrarily) one of these nodes as a *datum node*, i.e., as a reference for measuring electric potentials. By *connected*, we mean that any node can be reached from any other node in the circuit by traversing a path through the circuit elements. Note that the circuit in Fig. 3.2*b* is not connected. With respect to the chosen datum node, we define $n-1$ *node-to-datum* voltages as shown in Fig. 4.2. Since the circuit is a connected lumped circuit, these $n-1$ node-to-datum voltages are well-defined and, in principle, physically measurable quantities. Henceforth, we shall label them $e_1, e_2, \ldots, e_{n-1}$, and dispense with the + and − signs indicating the voltage reference direction. Note that $e_n = 0$ since node ⓝ is the chosen datum node.

Let v_{k-j} denote the voltage difference between node ⓚ and node ⓙ (see Fig. 4.2). Kirchhoff's voltage law states:

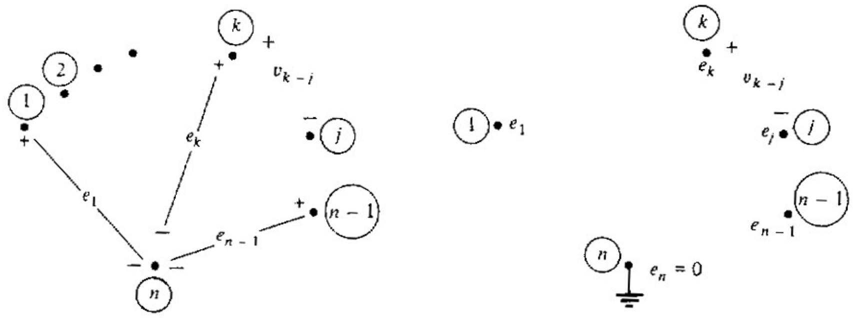

Figure 4.2 Labeling node-to-datum voltages for a circuit with n nodes.

> *KVL* For all lumped connected circuits, for all choices of datum node, for all times t, for all pairs of nodes ⓚ and ⓙ,
>
> $$v_{k-j}(t) = e_k(t) - e_j(t)$$

REMARK Clearly,

$$v_{j-k}(t) = e_j(t) - e_k(t) = -v_{k-j}(t) \tag{4.1}$$

Example The connected circuit in Fig. 4.3 is made of five 2-terminal elements and one 3-terminal element labeled T. There are five nodes, labeled ① through ⑤. Choosing (arbitrarily) node ⑤ as datum, we define the four node-to-datum voltages, e_1, e_2, e_3, and e_4. Therefore by KVL, we may write the following seven equations[5] (for convenience, we drop the dependence on t):

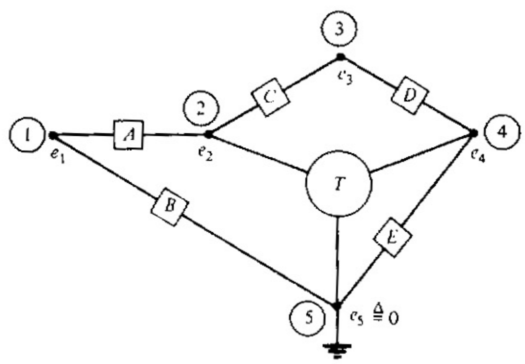

Figure 4.3 A connected circuit with five nodes.

[5] In view of Eq. (4.1) there are altogether two out of five, i.e., $C_2^5 = 10$ nontrivial equations which can be written.

$$
\begin{aligned}
v_{1-5} &= e_1 - e_5 = e_1 \\
v_{1-2} &= e_1 - e_2 \\
v_{2-3} &= e_2 - e_3 \\
v_{3-4} &= e_3 - e_4 \\
v_{4-5} &= e_4 - e_5 = e_4 \\
v_{2-4} &= e_2 - e_4 \\
v_{5-2} &= e_5 - e_2 = -e_2
\end{aligned} \tag{4.2}
$$

Note that $v_{1\text{-}5}$ and v_{1-2} are the voltages across the two-terminal elements B and A, respectively; v_{2-4}, v_{4-5}, and v_{5-2} are the voltages across the node pairs ②, ④; ④, ⑤; and ⑤, ② of the three-terminal element T, respectively.

If we add the last three equations in (4.2), we find that

$$v_{4-5} + v_{2-4} + v_{5-2} = 0$$

Let us consider the *closed node sequence* ②–④–⑤–②. It is *closed* because the sequence starts and ends at the same node ②. Thus for this particular closed node sequence, the sum of the voltages is equal to zero.

Let us consider a different closed node sequence ①–②–③–④–⑤–①. From the first five equations of (4.2) and using Eq. (4.1), we find that

$$v_{1-2} + v_{2-3} + v_{3-4} + v_{4-5} + v_{5-1} = 0$$

The closed node sequence ①–②–③–④–⑤–① is identified as a *loop* in the circuit, i.e., it is a closed path starting from any node, traversing through *two-terminal elements*, and ending at the same node. The closed node sequence ②–④–⑤–② is not a loop, neither is the closed node sequence ②–③–⑤–②.

Exercise Show that for the closed node sequence ②–③–⑤–② the sum of the voltages, v_{2-3}, v_{3-5}, and v_{5-2} is equal to zero.

We can state KVL in terms of closed node sequences:

> *KVL* (*closed node sequences*) For all lumped connected circuits, for all closed node sequences, for all times t, the algebraic sum of all node-to-node voltages around the chosen closed node sequence is equal to zero.

Theorem KVL in terms of node voltages is equivalent to KVL in terms of closed node sequences.

PROOF

1. We assume that KVL in terms of node voltages holds. Consider any closed node sequence, say ⓐ–ⓑ–ⓒ–ⓓ–ⓐ, and write the algebraic sum of all voltages around that sequence.

$$v_{a-b} + v_{b-c} + v_{c-d} + v_{d-a}$$

   By KVL in terms of node voltages this sum can be expressed as

$$(e_a - e_b) + (e_b - e_c) + (e_c - e_d) + (e_d - e_a) = 0$$

   so the first statement implies the second.

2. Now assume that KVL in terms of closed node sequences is true. Consider any closed node sequence, say ⓟ–ⓠ–ⓡ–ⓟ then

$$v_{p-q} + v_{q-r} + v_{r-p} = 0 \tag{4.3}$$

   Choosing (arbitrarily) ⓡ as the datum node, we have $v_{q-r} = e_q$ and $v_{r-p} = -e_p$ by definition of the node-to-datum voltages. Therefore from Eq. (4.3), we obtain

$$v_{p-q} = e_p - e_q$$

   So KVL in terms of closed node sequences implies KVL in terms of node voltages. ∎

REMARK For any given connected circuit with n nodes, let us choose (arbitrarily) node ⓝ as the datum node; then the $n-1$ node-to-datum voltages $e_1, e_2, \ldots, e_{n-1}$ specify uniquely and unambiguously the voltage v_{j-k} from any node ⓙ to any other node ⓚ in the circuit. This fact is of crucial importance in circuit theory and is the key concept in node analysis of Chap. 5.

4.3 Kirchhoff's Current Law (KCL)

A fundamental law of physics asserts that electric charge is conserved: There is no known experiment in which a net electric charge is either created or destroyed. Kirchhoff's current law (KCL) expresses this fundamental law in the context of lumped circuits.

To express KCL we shall use gaussian surfaces. A *gaussian surface* is by definition a two-sided "balloon-like" closed surface. Since it is two-sided, it has an "inside" and an "outside." To express the fact that the sum of the charges inside the gaussian surface $\mathcal{S}$ is constant, we shall require that at all times, the algebraic sum of all the currents leaving the surface $\mathcal{S}$ is equal to zero. Let us choose $\mathcal{S}$ so that it cuts only the connecting wires which connect the circuit elements as shown in Fig. 4.4. In the circuit, we have shown a four-terminal element: an operational amplifier, which is connected to the rest of the circuit