The two scalar equations (7.24) are linearly independent; furthermore, their coefficient matrix is in row echelon form. Thus if we consider i_3 and i_4 as parameters, Eq. (7.24) gives

$$\mathbf{i} = (i_1, i_2, i_3, i_4) \in K_i \Leftrightarrow \begin{cases} i_1 = -(i_3 + i_4) \\ i_2 = i_3 + i_4 \end{cases} \quad \left.\begin{matrix} i_3 \text{ and } i_4 \text{ are} \\ \text{arbitrary} \end{matrix}\right\} \tag{7.25}$$

The right-hand side of Eq. (7.25) is the general solution of Eq. (7.24). Physically, it asserts: Any $\mathbf{i} \in K_i$ is the superposition of an arbitrary loop current i_3 going through the loop $\{\beta_3, \beta_2, \beta_1\}$ and of an arbitrary loop current i_4 going through the loop $\{\beta_4, \beta_2, \beta_1\}$. K_i is clearly a two-dimensional subspace in $\mathbb{R}^4$.

KVL, namely $\mathbf{v} = \mathbf{A}^T\mathbf{e}$, reads in the present case

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} e_1 + \begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \end{bmatrix} e_2 \tag{7.26}$$

Thus K_v is the two-dimensional subspace formed by all linear combinations of the vectors $(1, 0, 1, 1)^T$ and $(0, 1, -1, -1)^T$.

Exercise Verify that for all i_3, i_4 and for all e_1, e_2, the $\mathbf{i}$ and the $\mathbf{v}$ given respectively by Eqs. (7.25) and (7.26) satisfy $\mathbf{v}^T\mathbf{i} = 0$, as expected by Tellegen's theorem.

PI

SUMMARY

- A physical circuit may be modeled by a *lumped* circuit provided that its physical dimension is small compared with the wavelength corresponding to the highest frequency of interest. For any *connected* lumped circuit, the voltage across any two nodes and the current anywhere in the circuit is well-defined. Kirchhoff's laws hold for any lumped circuit.
- KVL states for all lumped connected circuits, for all choices of datum node, for all times t, for all pairs of nodes ⓚ and ⓙ,

$$v_{k-j}(t) = e_k(t) - e_j(t)$$

  KVL can also be stated in terms of closed node sequences and in terms of loops.
- KCL states for all lumped circuits, for all gaussian surfaces $\mathcal{S}$, for all times t, the algebraic sum of all the currents leaving the gaussian surface $\mathcal{S}$ at time t is equal to zero. KCL can also be stated in terms of nodes and in terms of cut sets.

- Any circuit element can be represented by an element graph. A two-terminal element is represented by a digraph which contains two nodes and one branch. An n-terminal element is represented by a digraph with n nodes and $n-1$ branches. Therefore, there are $n-1$ branch voltages and $n-1$ branch currents for an n-terminal element.
- A two-port is represented by a digraph with four nodes and two unconnected branches. There are two independent branch voltages and two independent branch currents.
- Any lumped circuit can be represented by a circuit graph, called a digraph, which depicts the interconnection properties of the circuit and the chosen reference directions. For a connected digraph the node-branch incidence relation is given by a reduced incidence matrix $\mathbf{A}$ of $n-1$ rows and b columns, where n is the number of nodes and b is the number of branches.
- KCL: The b-dimensional branch current vector $\mathbf{i}$ is constrained by $\mathbf{Ai} = \mathbf{0}$.
- KVL: The branch voltage vector $\mathbf{v}$ and the node-to-datum voltage vector $\mathbf{e}$ are related by $\mathbf{v} = \mathbf{A}^T\mathbf{e}$.
- Tellegen's theorem: For any digraph, for all $\mathbf{v}$'s satisfying KVL, and for all $\mathbf{i}$'s satisfying KCL, $\mathbf{v}^T\mathbf{i} = 0$, provided that the associated reference directions are used for each branch.
- Kirchhoff's laws and Tellegen's theorem are valid for any lumped circuit regardless of the nature of the circuit elements. They reflect the interconnection properties of the circuit.

PROBLEMS

Digraph

1 Consider the circuit shown in Fig. P1.1. Choose the following datum for each device except D_2, which is a three-port device.

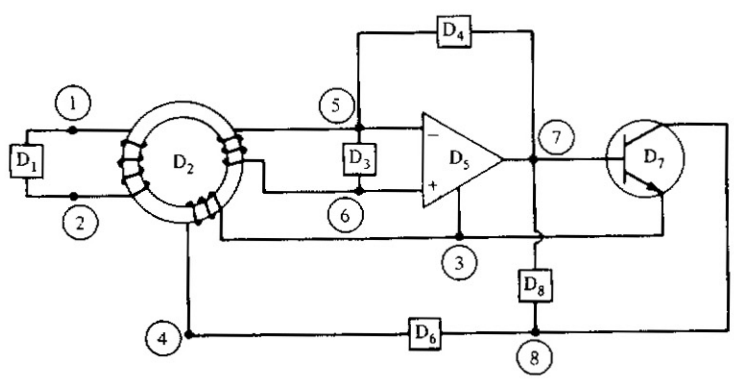

Figure P1.1

Device No.	Datum terminal no.
1	1
3	6
4	6
5	3
6	8
7	7
8	7

(*a*) Using the datum terminal specified, draw the element graph for each device in Fig. P1.1.

(*b*) Using the element graphs from (*a*), draw the digraph of the circuit in Fig. P1.1.

(*c*) Repeat (*b*) but with the datum terminal for the op amp changed to 7 and that of the transistor changed to 8.

2 Consider the circuit shown in Fig. P1.2. Choose the following datum for each device, except D_8, which is a three-port device.

(*a*) Draw the element graph with the datum terminal specified above for each device in Fig. P1.2.

(*b*) Using the element graphs from (*a*), draw the digraph for the circuit in Fig. P1.2.

(*c*) Repeat (*b*) but with the datum terminal for the transistor changed to 3, and that of the op amp changed to 8.

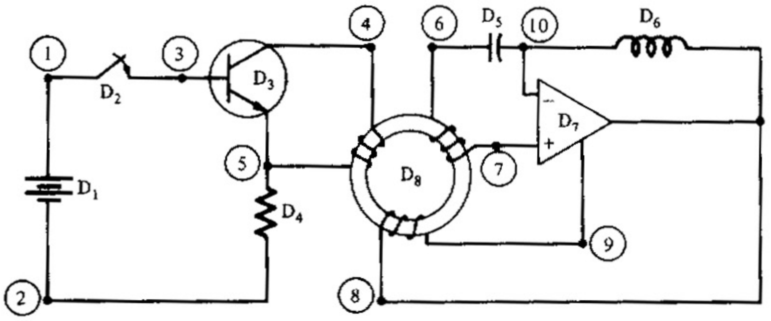

Figure P1.2

Device no.	Datum terminal no.
1	2
2	1
3	5
4	5
5	10
6	10
7	9

Digraph and KVL

3 Consider the circuit shown in Fig. P1.3.

(*a*) Given $v_{2-3} = 10$ V, $v_{6-3} = 6$ V, and $v_{4-1} = 2$ V, find v_{6-1}, v_{4-6}, and v_{4-2}.

(*b*) Draw the digraph with terminal 3 chosen as the datum of the op amp and terminal 4 chosen as the datum of the transistor. Repeat (*a*) using this digraph.

(*c*) Repeat (*b*) but with terminal ⑤ chosen as the datum terminal for both the op amp and the transistor.

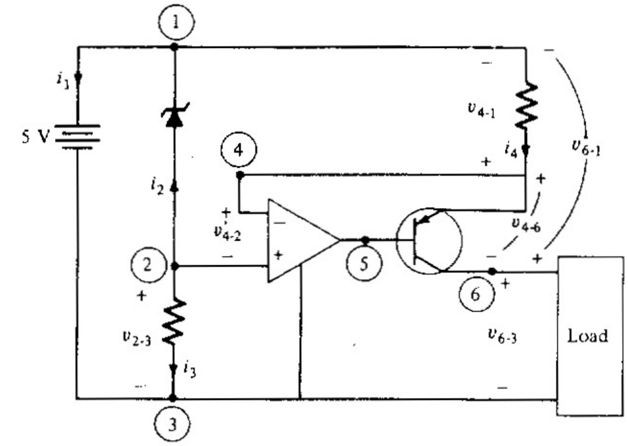

Figure P1.3

Digraph and closed node sequence

4 For the circuit shown in Fig. P1.4:

(*a*) Draw the digraph. Choose node ⑤ as the datum terminal of the op amp and use the associated reference convention.

(*b*) Write 10 KVL equations for each closed node sequence which contains four nodes.

(*c*) Are the equations from (*b*) linearly independent?

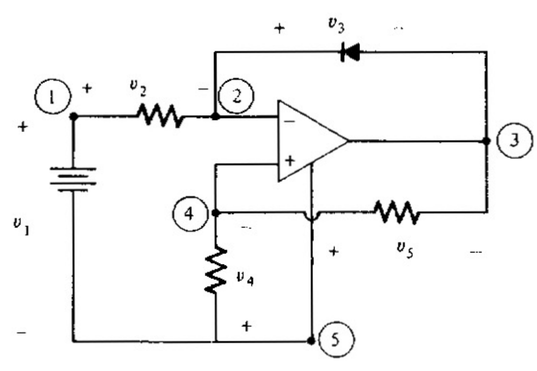

Figure P1.4

Incidence matrix, KVL, and KCL

5 (*a*) For the circuit shown in Fig. 4.4 in the text, draw the digraph by choosing node ④ as the datum node of the op amp.

(*b*) Determine the incidence matrix $\mathbf{A}_a$. What is the rank of $\mathbf{A}_a$?

(*c*) Choosing node ⑤ as the datum node, write down the KCL and KVL equations based on the incidence matrix $\mathbf{A}_a$ from (*b*).

Degree of freedom, KVL, and KCL

6 (*a*) For the circuit in Fig. 4.4 in the text, but using the digraph in Fig. 5.8, how many branch voltages can you specify independently? Specify such a set arbitrarily and demonstrate that the remaining voltages can be determined.

(*b*) Repeat (*a*) with branch currents.

(*c*) For the voltages and currents determined in (*a*) and (*b*), verify that Tellegen's theorem is valid.

Cut-set equations and linear independence

7 (*a*) For the digraph in Fig. 5.8, determine all KCL cut-set equations which are not included in the KCL node equations.

(*b*) Is the above set of equations linearly independent? If it is, prove it. If it is not, delete a minimum subset such that the remaining equations are linearly independent. Does the resulting set represent a maximal set, i.e., does it contain all information of the digraph?

Loop equations and linear independence

8 (*a*) For the digraph in Fig. 5.8, write KVL loop equations for all loops containing four or more branches.

(*b*) Repeat part (*b*) of Prob. 7 for the above loop equations.

Gaussian surface and cut sets

9 (*a*) Draw the digraph for the "full-wave rectifier circuit" shown in Fig. P1.9.

(*b*) Consider the following subsets of branches from this digraph: $\{1,2\}$, $\{1,2,3,4\}$, $\{4,5,6,7\}$, $\{3,6\}$, $\{3,4,5,6\}$, $\{1,2,4,5\}$, and $\{1,2,3,5,7\}$. Identify those subsets which qualify as branches cutting a gaussian surface and write the associated KCL equation. Explain why the remaining subsets do not qualify.

(*c*) Identify those subsets which qualify as cut sets and write the associated KCL equations. Explain why the remaining subsets do not qualify.

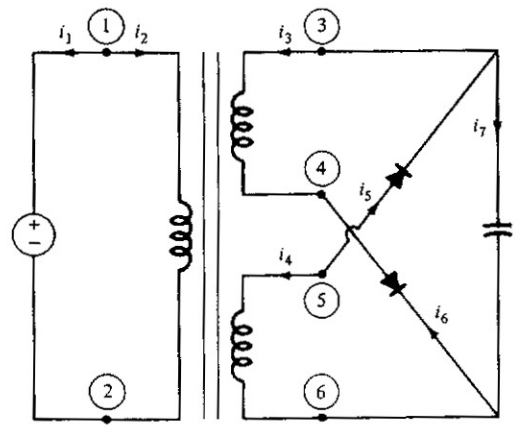

Figure P1.9

Incidence matrix and hinged graphs

10 (*a*) Write the incidence matrix $\mathbf{A}_a$ associated with the digraph from Prob. 9(*a*) and verify that the rows are not linearly independent. Why?

(*b*) Delete any row from $\mathbf{A}_a$ and verify that the remaining rows are still not linearly independent. Why?

(*c*) Hinge nodes ② and ⑥ and write the associated reduced incidence matrix **A**. Verify that the rows of **A** are linearly independent.

(*d*) Write a system of linearly independent KCL and KVL equations using **A**.

Incidence matrix and cut sets

11 For the digraph shown in Fig. P1.11:

(*a*) Write the incidence matrix $\mathbf{A}_a$.

(*b*) With node ③ as the datum node, write the KVL equations

$$\mathbf{v} = \mathbf{A}^T\mathbf{e}$$

(*c*) Give all the cut sets not already represented by $\mathbf{A}_a$.

(*d*) Select a maximum subset of the above which leads to linearly independent KCL equations.

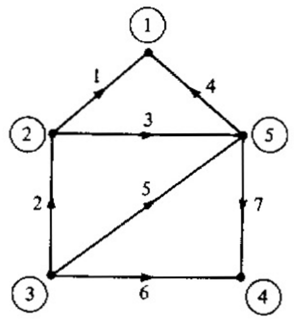

Figure P1.11

Degree of freedom, KVL, and KCL

12 In the circuit shown in Fig. P1.12, the branch current reference directions are indicated.

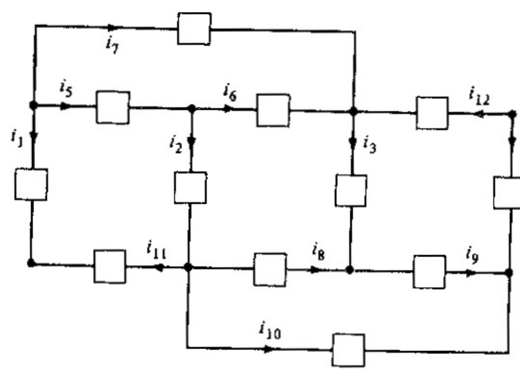

Figure P1.12

(*a*) How many branch currents can be independently specified? Why?

(*b*) If the following currents are given in amperes:

$$i_7 = -5 \qquad i_4 = 5 \qquad i_{10} = -3 \qquad i_3 = 1 \qquad i_1 = 24$$

is it possible to determine the remaining currents? Determine as many as you can.

(*c*) Let the branch voltages be measured in the associated reference directions. How many of the branch voltages can be independently specified? Why?

(*d*) If the following voltages are given in volts:

$$v_1 = 10 \qquad v_2 = 5 \qquad v_4 = -3 \qquad v_6 = 2 \qquad v_7 = -3 \qquad v_{12} = 8$$

determine as many branch voltages as possible.

Cut-set equations and linear independence

13 (*a*) Enumerate all cut sets for the digraph in Fig. P1.13.

(*b*) Write a KCL equation corresponding to each cut set from (*a*).

(*c*) Show that the equations from (*b*) are linearly dependent.

(*d*) Extract a subset of KCL equations from (*b*) containing a maximum number, ρ, of linearly independent equations. Verify that $\rho = n - p$, where n = number of nodes and p = number of connected components (or separate parts) in the digraph.

(*e*) Hinge nodes ⑤ and ⑦ and verify that the same KCL equations from (*b*) also hold for the resulting "connected" digraph.

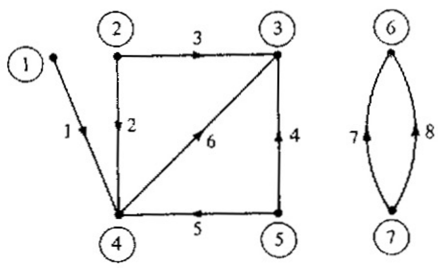

Figure P1.13

Incidence matrix, KVL, and KCL

14 (*a*) Write the incidence matrix $\mathbf{A}_a$ for the digraph in Fig. P1.14.

(*b*) Write the reduced incidence matrix **A** with node ⑤ as the datum node.

(*c*) Using **A**, write a system of linearly independent KVL and KCL equations.

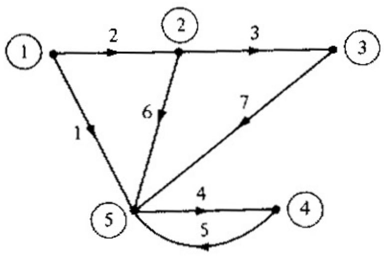

Figure P1.14

Degree of freedom, Tellegen's theorem

15 (*a*) Specify the degree of freedom (i.e., dimensions) of the KCL and KVL solution sets associated with the digraph from Fig. P1.14.

(*b*) Find two (among infinitely many) branch voltage distributions $\{v_1', v_2', \ldots, v_7'\}$ and $\{v_1'', v_2'', \ldots, v_7''\}$ which satisfy KVL for the digraph in Fig. P1.14 by assigning, arbitrarily, two distinct sets of node-to-datum voltages $\{e_1', e_2', e_3', e_4'\}$ and $\{e_1'', e_2'', e_3'', e_4''\}$, respectively.

(*c*) Find two (among infinitely many) branch current distributions $\{i'_1, i'_2, \ldots, i'_7\}$ and $\{i''_1, i''_2, \ldots, i''_7\}$ which satisfy KCL for the digraph in Fig. P1.14.

(*d*) Use your two sets of voltage and current distributions from (*b*) and (*c*) to verify Tellegen's theorem.

Incidence matrix and cut sets

16 Given the reduced incidence matrix of a digraph:

$$\begin{array}{rc}
 & \begin{array}{c} \text{Node} \end{array} \\
\mathbf{A} = & \left[\begin{array}{ccccc} 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right] \begin{array}{c} ① \\ ② \\ ③ \end{array} \\
\text{Branch} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array}
\end{array}$$

(*a*) Draw the associated digraph and mark all nodes and branches.

(*b*) List all cut sets which are not already included in **A**.

Incidence matrix, digraphs, cut-set and node equations

17 (*a*) Draw a digraph whose incidence matrix $\mathbf{A}_a$ is given by

$$\begin{array}{rc}
 & \text{Node} \\
\mathbf{A}_a = & \left[\begin{array}{cccccc} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{array}\right] \begin{array}{c} ① \\ ② \\ ③ \\ ④ \end{array} \\
\text{Branch} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}
\end{array}$$

(*b*) Given the following subgraphs of the digraph obtained in (*a*), identify the ones which form cut sets and the ones which are associated with gaussian surfaces.

$$\{1, 2, 3, 4\}, \ \{5, 6\}, \ \{2, 4, 5, 6\}, \ \{1, 3, 5, 6\}$$

(*c*) For the cut sets from (*b*), write down the corresponding KCL equations and also express these equations as the sum of appropriate node equations.

Cut sets, rank, and linear independence

18 A circuit has a digraph, as shown in Fig. P1.18.

(*a*) Choose node ④ as datum; write down the reduced incidence matrix **A**.

(*b*) Establish the rank of **A**.

(*c*) Show that the cut-set equation for the cut set $\{2, 3, 4, 5\}$ is linearly dependent on the node equations.

(*d*) Given $i_1 = 1\text{ A}$, $i_3 = 3\text{ A}$, and $i_5 = 5\text{ A}$, determine, if you can, all other currents.

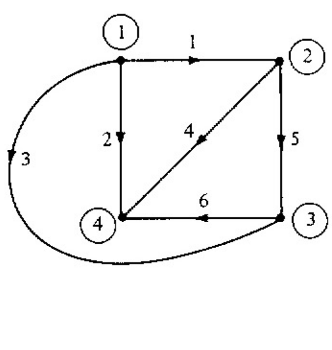

Figure P1.18

Loop equations and linear independence

19 A given circuit leads to the digraph of Fig. P1.19. For the digraph shown:

(*a*) List all possible loops and, for each one, write a KVL equation in terms of branch voltages.

(*b*) Are the equations from (*a*) linearly independent? If not, show that they are not.

(*c*) How many linearly independent loop equations are there in the list above? Justify your answer.

(*d*) Is it true that $i_7 + i_9 = 0$? Prove or disprove it.

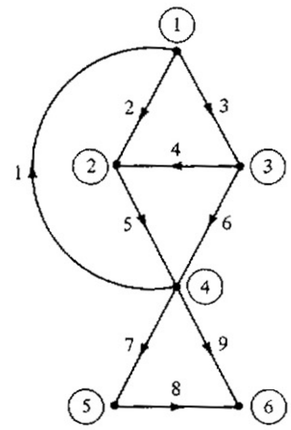

Figure P1.19

Tellegen's theorem

20 Let N be a two-port made of arbitrary interconnections of linear two-terminal linear resistors (i.e., each resistor satisfies Ohm's law: $v_j = R_j i_j$). Consider the following two experiments:

Experiment 1. Drive N with two voltage sources v_1' and v_2' and measure the resulting port currents i_1' and i_2', respectively. (See Fig. P1.20.)

Experiment 2. Drive N with two voltage sources v_1'' and v_2'' and measure the resulting port currents i_1'' and i_2'', respectively.

(*a*) Prove that the two sets of measurements are related as follows:

$$v_1' i_1'' + v_2' i_2'' = v_1'' i_1' + v_2'' i_2'$$

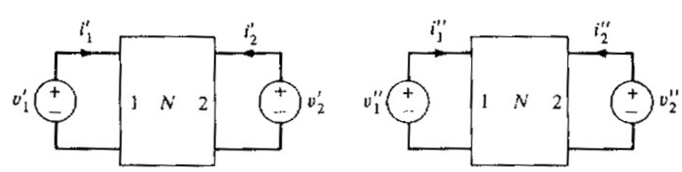

Figure P1.20

21 Let N be the two-port described in Prob. 20. Consider the following experiments

Experiment 1. Drive port 1 with a voltage source with voltage E and measure the current i_2' in the short circuit across port 2. (See Fig. P1.21.)

Experiment 2. Drive port 2 with a voltage source with an identical voltage E and measure the current i_1'' in the short circuit across port 1.

Prove that $i_2' = i_1''$, and state this remarkable property in words (reciprocity property).

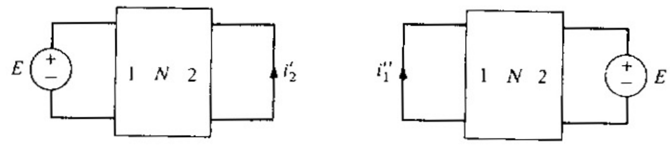

Figure P1.21

22 Without drawing the digraph, verify that Tellegen's theorem holds for any v_0 in the circuit shown in Fig. P1.22.

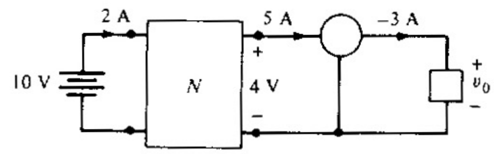

Figure P1.22

23 Let N denote a one-port made of n-terminal and n-port elements. Prove that the instantaneous power entering N at time t is equal to the sum of the instantaneous power entering each element inside N at time t.

24 In the circuits shown in Fig. P1.24, let v_k and i_k be the branch voltage and current in circuit A and $\hat{v}_k$ and $\hat{i}_k$ be the branch voltage and current in circuit B. The following measurements have been obtained:

$$i_s = 1\text{ A} \qquad \hat{v}_s = 3\text{ V} \qquad v_L = 2\text{ V}$$

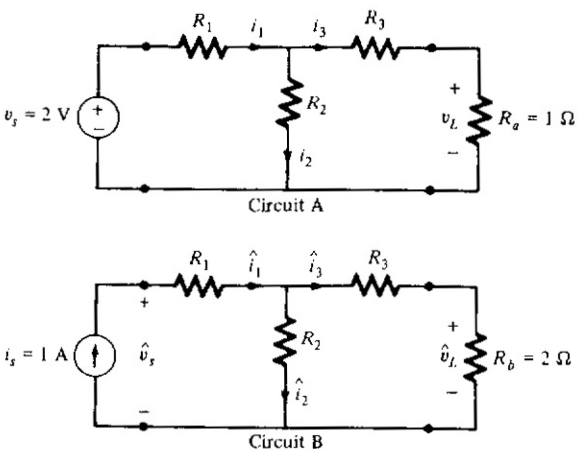

Figure P1.24

Use a particular form of Tellegen's theorem to determine $\hat{v}_L$, where R_1, R_2, and R_3 are unknown resistors satisfying Ohm's law.

25 Consider the circuit shown in Fig. P1.25. Two sets of measurements on this circuit give the following results:

(i) When $R_L = 2\,\Omega$: $v_1 = 8$ V $i_1 = -2$ A $v_L = 2$ V

(ii) When $R_L = 4\,\Omega$: $\hat{v}_1 = 12$ V $\hat{i}_1 = -2.4$ A $\hat{v}_L = ?$

Determine $\hat{v}_L$, given that R_1, R_2, R_3, and R_4 are linear resistors satisfying Ohm's law.

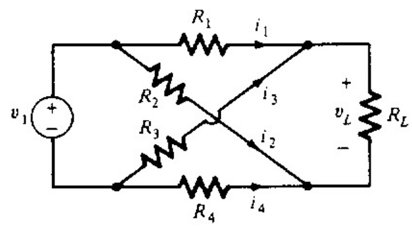

Figure P1.25