Before writing KCL on a cut set, we assign arbitrarily a positive reference direction by an arrowhead.

Some Examples of cut sets  

<table>
<thead>
<tr>
<th>Cut Set</th>
<th>KCL Cut Set Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>{3,5,7}</td>
<td>(-i_3 - i_5 - i_7 = 0)</td>
</tr>
<tr>
<td>{1,3,8}</td>
<td>(i_3 - i_1 - i_8 = 0)</td>
</tr>
<tr>
<td>{2,5,7,8}</td>
<td>(i_2 + i_5 + i_7 + i_8 = 0)</td>
</tr>
<tr>
<td>{3,4,5}</td>
<td>(i_3 - i_4 + i_5 = 0)</td>
</tr>
</tbody>
</table>
A Circuit with 3 different digraphs

1. Choose \( \Box \) as datum for \( D \)

2. Choose \( \Box \) as datum for \( D \)

3. Choose \( \Box \) as datum for \( D \)
• Circuits containing $n$-terminal devices can have many distinct digraphs, due to different (arbitrary) choices of the datum terminal for each $n$-terminal device.

• Although the KCL and KVL equations associated with 2 different digraphs of a given circuit are different, they contain the same information because each set of equations can be derived from the other.
KCL at $\mathbb{2}$ : $i_3 + i_4 = 0$

KCL at $\mathbb{4}$ : $i_5 + i_6 = 0$

KVL around $\mathbb{2}-\mathbb{3}-\mathbb{2}$ : $v_4 - v_3 = 0$

KVL around $\mathbb{4}-\mathbb{5}-\mathbb{4}$ : $v_6 - v_5 = 0$
Since nodes 3 and 5 are now the same node, they can be combined into one node, and the redrawn digraph is called a hinged graph.
Adding a wire connecting one node from each separate component does **not** change KVL or KCL equations.
Adding a wire connecting one node from each separate component does not change KVL or KCL equations.
Adding a wire connecting one node from each separate component does not change KVL or KCL equations.

\[ \{7\} \text{ is a cut set} \quad \Rightarrow \quad i_7 = 0 \]
Associated Reference Convention:

2-port Device

Device Graph

n-port Device
KCL at ①: \[ i_1 + i_2 - i_6 = 0 \]

KVL around ①-③-④-②-①: \[ v_2 + v_5 - v_4 - v_1 = 0 \]
These 3 KVL equations are not linearly-independent because the 3rd equation can be obtained by adding the first 2 equations:

\[(v_2 + v_3 - v_1) + (-v_3 + v_5 - v_4) = v_2 - v_1 + v_5 - v_4 = 0\]
Associated Reference Convention:

A current direction is chosen entering each positively-referenced terminal.

Device Graph: DIGRAPH (Directed Graph)
Circuit N

\[ V_1 = 2 \text{A}, v_2 = v_3, v_4 = 6V, 4\Omega, 3\Omega \]

\[ i_1, i_2, i_3, i_4 \]

Digraph G

Reduced Incidence Matrix A

<table>
<thead>
<tr>
<th>branch number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
### KCL:

\[
\begin{bmatrix}
1 & 1 & 0 & -1 \\
-1 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2 \\
i_3 \\
i_4
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\Rightarrow
\begin{aligned}
i_1 + i_2 - i_4 &= 0 \\
-i_1 + i_3 &= 0
\end{aligned}
\]

### KVL:

\[
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4
\end{bmatrix}
= 
\begin{bmatrix}
1 & -1 \\
1 & 0 \\
0 & 1 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2
\end{bmatrix}
\Rightarrow
\begin{aligned}
v_1 &= e_1 - e_2 \\
v_2 &= e_1 \\
v_3 &= e_2 \\
v_4 &= -e_1
\end{aligned}
\]
Circuit N

Number of nodes: \( n = 3 \)
Number of branches: \( b = 4 \)
Number of circuit variables: \( 2b+(n-1) = (2\times4)+(3-1) = 10 \)
Number of Independent KCL Equations: \( n-1 = 2 \)
Number of Independent KVL Equations: \( b = 4 \)
Total number of independent KCL and KVL Equations: \( b+(n-1) = 6 \)

We need “\( b \)” additional independent equations in order to obtain a system of \( 2b+(n-1) \) independent equations in \( 2b+(n-1) \) circuit variables.

The additional equations must come from the constitutive relation which relate the terminal voltages and currents of the circuit elements.

\[
\begin{align*}
\mathbf{e} &= \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \\
\mathbf{v} &= \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}, \\
\mathbf{i} &= \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}
\end{align*}
\]
Let us rearrange all 10 independent equations as follow:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
-1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -3 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2 \\
v_1 \\
v_2 \\
v_3 \\
v_4 \\
i_1 \\
i_2 \\
i_3 \\
i_4 \\
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 6 \\ 2 \end{bmatrix}
\]
Let us rearrange all 10 independent equations as follow:

\[
\begin{bmatrix}
0 & 0 & A \\
-A^T & 1 & 0 \\
0 & H_v & H_i \\
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2 \\
v_1 \\
v_2 \\
v_3 \\
v_4 \\
i_1 \\
i_2 \\
i_3 \\
i_4 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
6 \\
2 \\
\end{bmatrix}
\]
How Many Circuit Variables?

Answer: Total Number of Circuit Variables = $2b + n - 1$

Circuit N

Number of Nodes: $n = 3$

\[ e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \]

node-to-datum voltages: $n-1 = 2$

\[ v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \]

branch voltages: $b = 4$

\[ i = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} \]

branch currents: $b = 4$

\[ \therefore \text{There are } (n-1)+b+b = 10 \text{ circuit variables; namely, } \{e_1, e_2, v_1, v_2, v_3, v_4, i_1, i_2, i_3, i_4\}. \]
There are infinitely many sets of branch voltages \((v_1, v_2, v_3, v_4)\) which satisfy KVL for G.

2 Examples satisfying KVL:

- **KVL solution 1:** \(v_1 = -3V, v_2 = 2V, v_3 = 5V, v_4 = -2V\)
- **KVL solution 2:** \(\hat{v}_1 = 2V, \hat{v}_2 = 4V, \hat{v}_3 = 2V, \hat{v}_4 = -4V\)

There are infinitely many sets of branch currents \((i_1, i_2, i_3, i_4)\) which satisfy KCL for G.

2 Examples satisfying KCL:

- **KCL solution 1:** \(i_1 = 3A, i_2 = 2A, i_3 = 3A, i_4 = 5A\)
- **KCL solution 2:** \(\hat{i}_1 = 6A, \hat{i}_2 = -4A, \hat{i}_3 = 6A, \hat{i}_4 = 2A\)

**NOTE:** So far we have not specified what circuit elements are used in this circuit. This explains why the voltage and current solutions are **not** unique.
Example 1

KVL solution: Choose \( v_1 = -3V, v_2 = 2V, v_3 = 5V, v_4 = -2V \)

KCL solution: Choose \( i_1 = 3A, i_2 = 2A, i_3 = 3A, i_4 = 5A \)

\[
\sum_{j=1}^{4} v_j i_j = (-3)(3) + (2)(2) + (5)(3) + (-2)(5) = -9 + 4 + 15 - 10 = 0
\]
Example 2

KVL solution: Choose $\hat{v}_1 = 2V$, $\hat{v}_2 = 4V$, $\hat{v}_3 = 2V$, $\hat{v}_4 = -4V$

KCL solution: Choose $i_1 = 3A$, $i_2 = 2A$, $i_3 = 3A$, $i_4 = 5A$

$$\sum_{j=1}^{4} \hat{v}_j i_j = (2)(3) + (4)(2) + (2)(3) + (-4)(5)$$

$$= 6 + 8 + 6 - 20$$

$$= 0$$
Example 3

KVL solution: Choose \( v_1 = -3V, v_2 = 2V, v_3 = 5V, v_4 = -2V \)

KCL solution: Choose \( \hat{i}_1 = 6A, \hat{i}_2 = -4A, \hat{i}_3 = 6A, \hat{i}_4 = 2A \)

\[
\sum_{j=1}^{4} v_j \hat{i}_j = (-3)(6) + (2)(-4) + (5)(6) + (-2)(2) \\
= -18 - 8 + 30 - 4 \\
= 0
\]
Example 4

KVL solution: Choose \( \hat{\nu}_1 = 2V, \hat{\nu}_2 = 4V, \hat{\nu}_3 = 2V, \hat{\nu}_4 = -4V \)

KCL solution: Choose \( \hat{i}_1 = 6A, \hat{i}_2 = -4A, \hat{i}_3 = 6A, \hat{i}_4 = 2A \)

\[
\sum_{j=1}^{4} \hat{\nu}_j \hat{i}_j = (2)(6) + (4)(-4) + (2)(6) + (-4)(2)
\]

\[
= 12 - 16 + 12 - 8
\]

\[
= 0
\]
Solution:

\[ e_1 = 6V, \ e_2 = 6V \]
\[ v_1 = 0V, \ v_2 = 6V, \ v_3 = 6V, \ v_4 = -6V \]
\[ i_1 = 0A, \ i_2 = 2A, \ i_3 = 0A, \ i_4 = 2A \]

Verifying the solution satisfying Tellegen's Theorem:

\[
\sum_{j=1}^{4} v_j i_j = (v_1 i_1) + (v_2 i_2) + (v_3 i_3) + (v_4 i_4) \\
= (0)(0) + (6)(2) + (6)(0) + (-6)(2) \\
= 0 + 12 + 0 - 12 \\
= 0
\]
How to write An Independent System of KCL and KVL Equations

Let $N$ be any connected circuit and let the digraph $G$ associated with $N$ contain “$n$” nodes and “$b$” branches. Choose an arbitrary datum node and define the associated node-to-datum voltage vector $e$, the branch voltage vector $V$, and the branch current vector $i$. Then we have the following system of independent KCL and KVL equations.

\[(n-1) \text{ Independent KCL Equations :} \]

\[ A \mathbf{i} = \mathbf{0} \]

\[ b \text{ Independent KVL Equations :} \]

\[ \mathbf{v} = A^T e \]
Element Constitutive Relations

Element 1:  Resistor  
Described by Ohm’s Law :  \( v_1 = 4 \, i_1 \)

Element 2:  Resistor  
Described by Ohm’s Law :  \( v_2 = 3 \, i_2 \)

Element 3:  Voltage source  
Described by :  \( v_3 = 6 \)

Element 4:  Current source  
Described by :  \( i_4 = 2 \)

Rearranging these equations so that circuit variables appear on the left-hand side, we obtain

\[
\begin{align*}
    v_1 - 4 \, i_1 &= 0 \\
    v_2 - 3 \, i_2 &= 0 \\
    v_3 &= 6 \\
    i_4 &= 2
\end{align*}
\]

Observe we have obtained 4 additional independent equations.

Equations obtained from the element constitutive relations are guaranteed to be independent because different elements involved different circuit variables.
We can always recast any system of linear constitutive equations into the following standard matrix form

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -3 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
i_1 \\
i_2 \\
i_3 \\
i_4 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
6 \\
2 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
v \\
i \\
\end{bmatrix}
\text{independent source vector}
\]

\[
\mathbf{H}_v \mathbf{v} + \mathbf{H}_i \mathbf{i} = \mathbf{u}
\]
KCL Equations:

1. \( i_1 + i_2 - i_6 = 0 \)
2. \( -i_1 - i_3 + i_4 = 0 \)
3. \( -i_2 + i_3 + i_5 = 0 \)

\[ A \mathbf{i} = 0 \implies \]

\|
\begin{array}{cccccc}
1 & 1 & 0 & 0 & 0 & -1 \\
-1 & 0 & -1 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 & 1 & 0 \\
\end{array} \|

= \|
\begin{array}{c}
0 \\
0 \\
0 \\
\end{array} \|
KCL Equations:

1. \( i_1 + i_2 - i_6 = 0 \)
2. \( -i_1 - i_3 + i_4 = 0 \)
3. \( -i_2 + i_3 + i_5 = 0 \)

\[ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

\( A \) is called the reduced Incidence Matrix of the diagraph \( G \) relative to datum node \( 4 \).
KCL Node Equations:

Node No.  

1. \( i_1 + i_2 - i_6 = 0 \)
2. \( -i_1 - i_3 + i_4 = 0 \)
3. \( -i_2 + i_3 + i_5 = 0 \)
4. \( -i_4 - i_5 + i_6 = 0 \)

These 4 equations are linearly-dependent.

Matrix Formulation:

<table>
<thead>
<tr>
<th>Node No.</th>
<th>Branch no.</th>
<th>( \begin{bmatrix} i_1 \ i_2 \ i_3 \ i_4 \ i_5 \ i_6 \end{bmatrix} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 2 3 4 5 6</td>
<td>[ \begin{bmatrix} 1 &amp; 1 &amp; 0 &amp; 0 &amp; 0 &amp; -1 \end{bmatrix} ]</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>[ \begin{bmatrix} -1 &amp; 0 &amp; -1 &amp; 1 &amp; 0 &amp; 0 \end{bmatrix} ]</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>[ \begin{bmatrix} 0 &amp; -1 &amp; 1 &amp; 0 &amp; 1 &amp; 0 \end{bmatrix} ]</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>[ \begin{bmatrix} 0 &amp; 0 &amp; 0 &amp; -1 &amp; -1 &amp; 1 \end{bmatrix} ]</td>
</tr>
</tbody>
</table>

\[ \mathbf{A}_a \mathbf{i} = \mathbf{0} \]

**Incidence Matrix**

\[ a_{jk} = \begin{cases} 
1 & \text{if branch } k \text{ leaves node } j \\
-1 & \text{if branch } k \text{ enters node } j \\
0 & \text{if branch } k \text{ is not connected to node } j 
\end{cases} \]
KCL Equations:

1. \( i_1 + i_2 - i_6 = 0 \)
2. \(-i_1 - i_3 + i_4 = 0 \)
3. \(-i_2 + i_3 + i_5 = 0 \)

KVL Equations:

\[
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
v_5 \\
v_6
\end{bmatrix} =
\begin{bmatrix}
1 & -1 & 0 \\
1 & 0 & -1 \\
0 & -1 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
-1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2 \\
e_3
\end{bmatrix}
\]

\[
\begin{align*}
v_1 &= e_1 - e_2 \\
v_2 &= e_1 - e_3 \\
v_3 &= e_3 - e_2 \\
v_4 &= e_2 \\
v_5 &= e_3 \\
v_6 &= -e_1
\end{align*}
\]

Since \( v_j \) is present only in the \( j \)th equation, these \( k \) equations are linearly-independent.
Theorem

\[ A_i = 0 \]

gives the maximum possible number of linearly-independent KCL equations for a connected circuit.
Reduced Incidence Matrix

Let $G$ be a connected digraph with “$n$” nodes and “$b$” branches. Let $A_a$ be the Incidence Matrix of $G$. The $(n-1) \times b$ matrix $A$ obtained by deleting any one row of $A_a$ is called a Reduced-Incidence Matrix of $G$. 
Observation: The 4 KCL node equations are *not* linearly independent.

Adding the left side of the 4 KCL node equations, we obtain:

\[
(i_1 + i_2 - i_6) + (-i_1 - i_3 + i_4) + (-i_2 + i_3 + i_5) + (-i_4 - i_5 + i_6) = 0
\]

This means we can derive any one of these 4 equations from the other 3.

Example: Derive KCL equations at node ④:

Adding the first 3 node equations gives:

\[
(i_1 + i_2 - i_6) + (-i_1 - i_3 + i_4) + (-i_2 + i_3 + i_5) = i_4 + i_5 - i_6
\]
Reduced Incidence Matrix A

Let $G$ be a connected digraph with “$n$” nodes and “$b$” branches, the reduced incidence matrix $A$ relative to datum node ① is an $(n-1) \times b$ matrix whose coefficients $a_{jk}$ are obtained from the $(n-1)$ KCL equations written at the $n-1$ non-datum nodes:

$$a_{jk} = \begin{cases} 
1 & \text{if branch } k \text{ leaves node } j \\
-1 & \text{if branch } k \text{ enters node } j \\
0 & \text{if branch } k \text{ is not connected to node } j
\end{cases}$$
By applying the various versions of KCL, we can write many different KCL equations for each circuit. However, these equations are usually **not** linearly independent in the sense that each equation can be derived by a linear combination of the others.

How can we write a maximum set of **linearly-independent** KCL equations?
Simplest Method
to write linearly-Independent KCL Equations.

Given a connected circuit with “$n$” nodes, choose an arbitrary node as **datum**. Write a KCL equation at each of the remaining $(n-1)$ nodes.
Relationship between $\mathbf{A}$ and $\mathbf{A}_a$

Let $\mathbf{A}_a$ be the $n \times b$ Incidence matrix of a connected digraph $\mathcal{G}$ with “$n$” nodes and “$b$” branches.

By deleting any row corresponding to node $m$ from $\mathbf{A}_a$, we obtain the reduced incidence matrix $\mathbf{A}$ of $\mathcal{G}$ relative to the datum node $m$. 
Choose node 4 as datum node for digraph G

KCL Equations:

1. \( i_1 + i_2 - i_6 = 0 \)
2. \( -i_1 - i_3 + i_4 = 0 \)
3. \( -i_2 + i_3 + i_5 = 0 \)

Independent KCL Equations

\[
\mathbf{A} \mathbf{i} = \mathbf{0} \implies \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & -1 \\
-1 & 0 & -1 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
i_1 \\
i_2 \\
i_3 \\
i_4 \\
i_5 \\
i_6
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

Independent KVL Equations

\[
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
v_5 \\
v_6
\end{bmatrix} = \begin{bmatrix}
1 & -1 & 0 \\
1 & 0 & -1 \\
0 & -1 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
-1 & 0 & 0
\end{bmatrix} \begin{bmatrix}
e_1 \\
e_2 \\
e_3
\end{bmatrix} \implies \begin{align*}
v_1 &= e_1 - e_2 \\
v_2 &= e_1 - e_3 \\
v_3 &= -e_2 + e_3 \\
v_4 &= e_2 \\
v_5 &= e_3 \\
v_6 &= -e_1
\end{align*}
\]
choose \( \hat{e}_3 \) as datum and let \( \hat{e}_1, \hat{e}_2, \hat{e}_4 \) be new node-to-datum voltages.

**KCL Equations:**

\[
\begin{align*}
1 & \quad i_1 + i_2 - i_6 = 0 \\
2 & \quad -i_1 - i_3 + i_4 = 0 \\
4 & \quad -i_4 - i_5 + i_6 = 0
\end{align*}
\]

**Independent KCL Equations**

\[
\hat{A} i = 0 \quad \Rightarrow \quad \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

**Independent KVL Equations**

\[
\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \\ \hat{e}_4 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} v_1 = \hat{e}_1 - \hat{e}_2 \\ v_2 = \hat{e}_1 \\ v_3 = -\hat{e}_2 \\ v_4 = \hat{e}_2 - \hat{e}_4 \\ v_5 = -\hat{e}_4 \\ v_6 = -\hat{e}_1 + \hat{e}_4 \end{bmatrix}
\]
We can always recast any system of linear constitutive equations into the following standard matrix form:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
6 \\
2 \\
\end{bmatrix}
\]

\[H_v \mathbf{v} + H_i \mathbf{i} = \mathbf{u}\]
KCL \[
\begin{align*}
i_1 + i_2 - i_4 &= 0 \\
-i_1 + i_3 &= 0
\end{align*}
\]

KVL \[
\begin{align*}
v_1 &= e_1 - e_2 \\
v_2 &= e_1 \\
v_3 &= e_2 \\
v_4 &= -e_1
\end{align*}
\]

Element Constitative Relation \[
\begin{align*}
v_1 &= 4i_1 \\
v_2 &= 3i_2 \\
v_3 &= 6 \\
i_4 &= 2
\end{align*}
\]

We can always find the solution using Cramer’s rule.
For simple circuits, we can often find the solution by *as hoc* elimination and substitution of variables:

**EXAMPLE:**

- (5) and (9) \( \Rightarrow \) \( e_2 = 6 \) \( (11) \)
- (1) and (10) \( \Rightarrow \) \( i_1 + i_2 = 2 \) \( (12) \)
- (3), (7) and (11) \( \Rightarrow \) \( i_1 = \frac{1}{4}(e_1 - 6) \) \( (13) \)
- (4) and (8) \( \Rightarrow \) \( i_2 = \frac{1}{3}e_1 \) \( (14) \)

Substituting (10), (11), (12), and (13) into (1), we obtain

\[
\frac{1}{4}(e_1 - 6) + \frac{1}{3}e_1 - 2 = 0
\]

\( \Rightarrow \) \( e_1 = 6 \) \( (15) \)

**Complete Solution:**

\[\begin{align*}
e_1 &= 6 \text{V}, \quad e_2 = 6 \text{V} \\
v_1 &= 0 \text{V}, \quad v_2 = 6 \text{V}, \quad v_3 = 6 \text{V}, \quad v_4 = -6 \text{V} \\
i_1 &= 0 \text{A}, \quad i_2 = 2 \text{A}, \quad i_3 = 0 \text{A}, \quad i_4 = 2 \text{A}
\end{align*}\]

**Verification of solution via Tellegen’s Theorem**

\[
\sum_{j=1}^{4} v_j i_j = (v_1 i_1) + (v_2 i_2) + (v_3 i_3) + (v_4 i_4)
\]

\[= (0)(0) + (6)(2) + (6)(0) + (-6)(2) = 0\]
Tellegen’s Theorem

Let \( G \) be a diagraph with "b" branches.
Let \((v_1, v_2, \cdots, v_b)\) be any set of \( b \) voltages of \( G \) which satisfy KVL.
Let \((i_1, i_2, \cdots, i_b)\) be any set of \( b \) currents of \( G \) which satisfy KCL.

Then
\[
\sum_{j=1}^{b} v_j i_j = 0
\]

Proof: suppose
\[
v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_b \end{bmatrix}
\]
for \( G \) satisfies KVL, \( i = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_b \end{bmatrix} \) satisfies KCL

Then
\[
\sum_{j=1}^{b} v_j i_j = v^T i = (A^T e)^T i = e^T (A i) = 0
\]

where \( e = [e_1, e_2, \cdots, e_{n-1}]^T \) is any node-to-datum voltage.

Warning: By definition of a diagraph, each branch voltage \( v_j \) and branch current \( i_j \) associated with branch \( j \) must follow the Associated reference convention: \( i_j \) flows from the positive terminal to the negative terminal.
Suppose we choose:

- $i_j = 0$, if $i_j$ is not in loop "l"
- $i_j = 1$, if $i_j$ is in loop "l" and flows in the same direction as loop "l"
- $i_j = -1$, if $i_j$ is in loop "l" and flows in opposite direction as loop "l"

This choice of \{i_1, i_2, \ldots, i_b\} Satisfies KCL

\[
\sum_{j=1}^{b} v_j i_j = 0 \quad \text{(because } v_j \text{ chosen earlier satisfies Tellegen's theorem)}
\]

\[
0 = \sum_{j=1}^{b} v_j i_j = \sum_{b_j \text{ belonging to loop } "l"} v_j i_j + \sum_{b_j \text{ not belonging to loop } "l"} v_j i_j
\]

equals 0 because $i_j = 0$

\[
\Rightarrow \sum_{b_j \text{ belonging to loop } "l"} v_j i_j = 0 \quad \Rightarrow \quad \text{KVL}
\]
Relationship Between Kirchhoff’s Laws and Tellegen’s Theorem

1. **KCL** and **KVL** ➞ Tellegen’s Theorem

2. **KVL** and **Tellegen’s Theorem** ➞ **KCL**

3. **Tellegen’s Theorem** and **KCL** ➞ **KVL**
**Proof.**

Let \( v \) satisfy KVL for \( G \):

\[
v = A^T e \quad (1)
\]

Let \( v \) and \( i \) satisfy Tellegen’s Theorem:

\[
v^T i = 0 \quad (2)
\]

Substitute (1) for \( v \) in (2):

\[
(A^T e)^T i = 0 \quad (3)
\]

\[
e^T (A i) = 0 \quad (4)
\]

Since (4) is true for any node-to-datum voltages \( e \neq 0 \), (4) can be true only if

\[
(A i) = 0 \quad \Rightarrow \quad \text{KCL}
\]

\[\blacksquare\]
Tellegen’s Theorem and KCL $\rightarrow$ KVL

**Proof.**

Let $G$ be any connected digraph with $b$ branches $\{1, 2, \ldots, b\}$.

Let $\{i_1, i_2, \ldots, i_b\}$ be any set of branch currents satisfying KCL.

Choose *any* subset $\{b_a, b_b, \ldots, b_n\}$ of the $b$ branches which form a closed loop “$l$”. Let $\{v_1, v_2, \ldots, v_b\}$ be *any* set of branch voltages which, together with $\{i_1, i_2, \ldots, i_b\}$ satisfy Tellegen’s Theorem.

Our goal is to prove that the subset of these voltages which belong to the above closed loop “$l$” must satisfy KVL around the loop.
Applying Tellegen’s Theorem to Circuits Containing \((n+1)\)-terminal devices

Let \(N\) be any circuit containing \((n+1)\)-terminal devices.

*Step 1.* Assign a datum to each device. Assign “\(n\)” terminal-to-datum voltages for each \((n+1)\)-terminal device, following associated reference convention.

*Step 2.* Draw the digraph \(G\) of \(N\).

*Step 3.* Apply Tellegen’s theorem to \(G\).
Remarks
Tellegen’s theorem can be applied directly to a circuit provided we use Associated Reference convention for all device terminal currents and voltages.

\[
\sum_{j=1}^{4} v_j i_j = v_1 i_1 + v_2 i_2 + v_3 i_3 + v_4 i_4 = 0
\]

(choose 3 as datum node for the 3-terminal device)
Voltage and Current Solutions are Orthogonal!

Reduced Incidence Matrix

\[ A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \]

KCL: \[ A \mathbf{i} = 0 \]

\[ \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ i_1 + i_2 + i_3 = 0 \]
Voltage and Current Solutions are Orthogonal!

**Reduced Incidence Matrix**

\[ A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \]

\[ \text{KVL} : \mathbf{v} = A^T \mathbf{e} \]

\[ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \]

\[ v_1 = e_1, \quad v_2 = e_1, \quad v_3 = e_1 \]
Geometrical Interpretation of Tellegen’s Theorem

KCL : \( i_1 + i_2 + i_3 = 0 \)

KVL : \( v_1 = v_2 = v_3 \)

\[
\sum_{j=1}^{3} v_j i_j = v_1 i_1 + v_2 i_2 + v_3 i_3 = e_1 i_1 + e_1 i_2 + e_1 i_3 = e_1 (i_1 + i_2 + i_3) = 0
\]

All voltage solutions \((v_1, v_2, v_3)\) falling on this line satisfy KVL.

All current solutions \((i_1, i_2, i_3)\) falling on this plane satisfy KVL.