

Figure 5.14 (a) A "grounded" two-port and (b) its element graph.

5.4 Cut Sets and KCL

A very useful graph-theoretic concept is the cut set. Given a connected digraph \mathcal{G} , a set of branches \mathcal{C} of \mathcal{G} is called a cut set iff (a) the removal of all the branches of the cut set results in an unconnected digraph, which means that the resulting digraph is no longer connected, and (b) the removal of all but any one branch of \mathcal{C} leaves the digraph connected. Stated in another way, (b) implies that if any branch in the set is left intact, the digraph remains connected.

For the digraph of Fig. 5.15, $\mathscr{C}_1 = \{\beta_1, \beta_3\}$, $\mathscr{C}_2 = \{\beta_4, \beta_5, \beta_6\}$, and $\mathscr{C}_3 = \{\beta_4, \beta_5, \beta_7\}$ form cut sets. Here, β_k denotes "branch k."

Exercise Refer to Fig. 5.15.

- (a) Is $\{\beta_1, \beta_3, \beta_4, \beta_5, \beta_6\}$ a cut set?
- (b) List all cut sets of the digraph shown in Fig. 5.15.

REMARKS

- 1. Any cut set creates a partition of the set of nodes in the graph into two subsets.
- 2. To any cut set corresponds a gaussian surface which cuts precisely the same branches.
- 3. Similarly, to any gaussian surface corresponds either one cut set or a union of cut sets (see \mathcal{S}_1 in Fig. 5.15).
- 4. To each cut set we can define arbitrarily a *reference direction*, as shown by the arrows attached to the cut sets in Fig. 5.15.

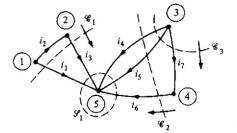


Figure 5.15 Digraph illustrating cut sets.

^{7 &}quot;iff" means "if and only if."

KCL (cut-set law) For all lumped circuits, for all time t, the algebraic sum of the currents associated with any cut set is equal to zero.

Example For the digraph shown in Fig. 5.16, the cut set $\mathscr{C} = \{\beta_1, \beta_2, \beta_3\}$ is indicated by the dashed line cutting through these branches. Let us assign a reference direction to \mathscr{C} as shown by the arrow; then the KCL applied to \mathscr{C} gives

$$i_1(t) + i_2(t) - i_3(t) = 0$$

The $-i_3$ comes about because the reference direction of i_3 disagrees with the reference direction of the cut set \mathscr{C} .

By now we have learned three forms of KCL, namely, in terms of (1) gaussian surfaces, (2) nodes, and (3) cut sets.

KCL theorem The three forms of the KCL are equivalent. Symbolically, 8

$$\binom{\text{KCL}}{\text{gaussian surface}} \Leftrightarrow \binom{\text{KCL}}{\text{node law}} \Leftrightarrow \binom{\text{KCL}}{\text{cut sets}}$$

PROOF

(1) \Rightarrow (2) Simply use a gaussian surface that surrounds only the node in question. For example, consider node 5 in Fig. 5.15: For gaussian surface \mathscr{S}_1 , KCL applied to \mathscr{S}_1 is identical with KCL applied to node 5, namely,

$$-i_1 - i_3 - i_4 - i_5 - i_6 = 0$$

(2) \Rightarrow (3) Any cut set partitions the set of nodes into two subsets. Writing the KCL equation for each node in such a subset and adding the results, we obtain the cut-set equation, except for maybe a -1 factor. For example, consider the cut set \mathscr{C}_2 in Fig. 5.15: If we

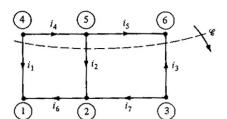


Figure 5.16 Digraph illustrating the reference direction of a cut set.

^{8 ⇒} means "implies"; ← means "is implied by"; ⇔ means "is equivalent to."

add the KCL equations applied to nodes 3 and 4, we obtain

$$i_4 + i_5 + i_6 = 0$$

(note that i_7 cancels out in the addition!), which is the cut-set equation for \mathscr{C}_2 .

(3) \Rightarrow (1) It is easy to demonstrate that the set of branches cut by a gaussian surface is either a cut set or a disjoint union of cut sets. So given any gaussian surface, let us write the KCL equation for each of these cut sets; then adding or subtracting these equations, we obtain the KCL equation for the gaussian surface. For example, consider gaussian surface \mathcal{S}_1 of Fig. 5.15. It is the union of cut set $\{\beta_1, \beta_3\}$ and cut set $\{\beta_4, \beta_5, \beta_6\}$ whose equations are, respectively,

$$-i_1 - i_3 = 0$$
$$+i_4 + i_5 + i_6 = 0$$

Subtracting the second equation from the first gives

$$-i_1 - i_3 - i_4 - i_5 - i_6 = 0$$

which is the KCL equation for gaussian surface \mathcal{G}_1 .

6 MATRIX FORMULATION OF KIRCHHOFF'S LAWS

6.1 Linear Independence

Consider a set of m linear algebraic equations in n unknowns: For j = 1, 2, ..., m

$$f_i(x_1, x_2, \dots, x_n) = \alpha_{i1}x_1 + \alpha_{i2}x_2 + \dots + \alpha_{in}x_n = 0$$
 (6.1)

where the α_{jk} 's are real or complex numbers. It is important to decide whether or not each equation brings new information not contained in the others; equivalently, it is important to decide whether the equations are linearly independent. These m equations are said to be *linearly dependent* iff there are constants k_1, k_2, \ldots, k_m and *not all zero* such that

$$\sum_{j=1}^{m} k_j f_j(x_1, x_2, \dots, x_n) = 0 \quad \text{for all } x_1, x_2, \dots, x_n$$
 (6.2)

Clearly if these m equations are linearly dependent, then at least one equation may be written as a linear combination of the others; in other words, that equation repeats the information contained in the others!

It is crucial to note that the left-hand side of Eq. (6.2) must be zero for all values of x_1, x_2, \ldots, x_n .

Example Consider an example where m = 3 and n = 4:

$$x_1 - x_2 + x_3 + 3x_4 = 0$$
$$2x_1 + 3x_2 - x_3 - 4x_4 = 0$$
$$-4x_1 - 11x_2 + 5x_3 + 18x_4 = 0$$

Direct calculation shows that with $k_1 = 2$, $k_2 = -3$, and $k_3 = -1$ the condition for Eq. (6.2) holds; in other words, these three equations are linearly dependent.

The set of m linear algebraic equations (6.1) is said to be *linearly independent* iff it is not linearly dependent.

In practice, we use gaussian elimination to decide whether or not a given set of linear equations is linearly dependent.

6.2 Independent KCL Equations

For a given circuit, we can write many KCL equations by the node law, the cut-set law, or using gaussian surfaces. How many of them are linearly independent and how to write a complete set that contains all the necessary information as far as KCL is concerned are the subjects of this subsection. We will give a systematic treatment by means of the digraph of the circuit under consideration: in particular, a list of nodes, a list of branches, and for each branch the specification of the node it leaves and of the node it enters. This is done by the *incidence matrix* \mathbf{A}_a of the digraph.

Let digraph \mathcal{G} have n nodes and b branches, then A_a has n rows—one row to each node—and b columns—one column to each branch. To see how the matrix is built up consider the four-node six-branch digraph shown in Fig. 6.1. Let us write the KCL equations for each node:

$$i_{1} + i_{2} - i_{6} = 0$$

$$-i_{1} - i_{3} + i_{4} = 0$$

$$-i_{2} + i_{3} + i_{5} = 0$$

$$-i_{4} - i_{5} + i_{6} = 0$$
(6.3)

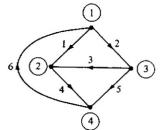


Figure 6.1 A digraph with four nodes and six branches.

In matrix form it reads

The 4×6 matrix just obtained is called the *incidence matrix* A_a of \mathcal{G} .

Exercise

- (a) Demonstrate that the four equations in (6.3) are linearly dependent.
- (b) Demonstrate that any three of the four equations in (6.3) are linearly independent.

In general, for an *n*-node *b*-branch connected digraph \mathcal{G} which does not contain self-loops the matrix \mathbf{A}_a is specified as follows: For i = 1, 2, ..., n and k = 1, 2, ..., b

$$a_{ik} = \begin{cases} +1 \text{ if branch } k \text{ leaves node } (i) \\ -1 \text{ if branch } k \text{ enters node } (i) \\ 0 \text{ if branch } k \text{ does not touch node } (i) \end{cases}$$
(6.5)

and the n node equations of \mathcal{G} read

$$\mathbf{A}_{a}\mathbf{i} = \mathbf{0} \tag{6.6}$$

where $\mathbf{i} = (i_1, i_2, \dots, i_b)^T$ is called the branch current vector.

REMARK Each column of A_a has precisely a single +1 and a single -1; consequently, if we add together the n equations in (6.6), all the variables i_1, i_2, \ldots, i_b cancel out; equivalently the n KCL equations are linearly dependent.

Suppose that for the *connected* digraph \mathcal{G} we choose a datum node and we throw away the corresponding KCL equation, then the remaining n-1 equations are linearly independent. Since this is important we state it formally:

Independence property of KCL equations For any connected digraph \mathcal{G} with n nodes, the KCL equations for any n-1 of these nodes form a set of n-1 linearly independent equations.

⁹ The digraph of circuits containing multiterminal elements will contain self-loops whenever one or more terminals are connected to the datum, as in the last exercise of Sec. 5.2.

Proof We prove it by contradiction. Suppose that the first k of these n-1 equations are *linearly dependent*. More precisely, there are k real constants $\gamma_1, \gamma_2, \ldots, \gamma_k$, not all zero, such that

$$\sum_{j=1}^{k} \gamma_j f_j(i_1, i_2, \dots, i_n) = 0 \quad \text{for all } i_1, i_2, \dots, i_n$$
 (6.7)

Without loss of generality, we may assume that $\gamma_j \neq 0$ for j = 1, 2, ..., k, i.e., there are exactly k equations in the sum of Eq. (6.7).

Consider the two sets of nodes in \mathcal{G} , namely, the set which corresponds to the k equations and that of the remaining nodes. Since the digraph is connected, there is at least one branch which connects a node in the first set to a node in the second set. Clearly the current in that branch appears only *once* in the first k node equations, hence that current cannot cancel out in the sum of Eq. (6.7). This contradiction shows that for any $k \le n - 1$ it is not the case that a subset of k of the KCL equations is linearly dependent. That is, these n - 1 equations are linearly independent.

If in A_a , the incidence matrix of the connected digraph \mathcal{G} , we delete the row corresponding to the datum node, we obtain the *reduced incidence matrix* A which is of dimension $(n-1) \times b$. The corresponding KCL equations read

$$\mathbf{Ai} = \mathbf{0} \tag{6.8}$$

As a consequence of the independence property just proved, we may state that the $(n-1) \times b$ matrix **A** is full rank, i.e., its n-1 rows are linearly independent vectors in the *b*-dimensional space. Stated in another way, (6.8) consists of n-1 linearly independent KCL equations.

6.3 Independent KVL Equations

Similarly, to write a set of complete linearly independent KVL equations in a systematic way is of crucial importance. Let us write KVL for the four-node six-branch digraph of Fig. 6.1. Using associated reference directions and choosing node ④ as the datum node, we obtain

$$v_{1} = e_{1} - e_{2}$$

$$v_{2} = e_{1} - e_{3}$$

$$v_{3} = -e_{2} + e_{3}$$

$$v_{4} = e_{2}$$

$$v_{5} = e_{3}$$

$$v_{6} = -e_{1}$$
(6.9)

or in matrix form

$$\mathbf{v} = \mathbf{Me} \tag{6.10}$$

where $\mathbf{v} = (v_1, v_2, \dots, v_b)^T$ is the branch voltage vector, $\mathbf{e} = (e_1, e_2, \dots, e_{n-1})^T$ is the node-to-datum voltage vector, and \mathbf{M} is a $b \times (n-1)$ matrix. Thinking in terms of KVL, we see that for $k = 1, 2, \dots, b$ and $i = 1, 2, \dots, n-1$

$$m_{ki} = \begin{cases} +1 \text{ if branch } k \text{ leaves node } (i) \\ -1 \text{ if branch } k \text{ enters node } (i) \\ 0 \text{ if branch } k \text{ does not touch node } (i) \end{cases}$$
(6.11)

Comparing Eq. (6.11) with (6.5), we conclude that

$$\mathbf{M} = \mathbf{A}^T$$

and more usefully, KVL is expressed by the equation

$$\mathbf{v} = \mathbf{A}^T \mathbf{e} \tag{6.12}$$

With a connected digraph \mathcal{G} A has n-1 linearly independent rows, and consequently \mathbf{A}^T has n-1 linearly independent columns.

REMARKS

- 1. Note that, in the digraph, (a) we choose current reference directions,
 - (b) we choose a datum node and define the reduced incidence matrix A,
 - (c) we write KCL as Ai = 0, (d) then we use associated reference directions to find that KVL reads $v = A^{T}e$. Thus whenever we invoke this last equation, we automatically use associated reference directions for the branch voltages. We also assume the same datum node is used in writing KCL and KVL.
- 2. When we deal with digraphs which are not connected, we could either use the concept of the hinged graph to make the digraph connected or treat each separate part independently. In the latter, each separate part will have its own incidence matrix and datum node.

7 TELLEGEN'S THEOREM

Tellegen's theorem is a very general and very useful theorem. We'll use it repeatedly in this text. Tellegen's theorem is a direct consequence of Kirchhoff's laws.

7.1 Theorem, Proof, and Remarks

Example Consider the digraph shown in Fig. 7.1. Choose arbitrarily the

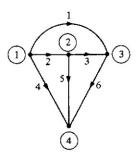


Figure 7.1 A digraph with four nodes and six branches.

values of the currents i_1 , i_2 , i_3 and calculate i_4 , i_5 , i_6 so that KCL is satisfied: Let

$$i_1 = 1$$
 $i_2 = 2$ $i_3 = 3$

hence

$$i_4 = -3$$
 $i_5 = -1$ $i_6 = 4$

Now choose arbitrarily v_4 , v_5 , and v_6 and calculate v_1 , v_2 , v_3 so that KVL is satisfied (note that we use associated reference directions). Let

$$v_4 = 4$$
 $v_5 = 5$ $v_6 = 6$

hence

$$v_1 = -2$$
 $v_2 = -1$ $v_3 = -1$

Note that i_1, i_2, \ldots, i_6 obey KCL and v_1, v_2, \ldots, v_6 obey KVL for the circuit under consideration. Now it is easy to verify that

$$\sum_{k=1}^{6} v_k i_k = 0$$

This result is surprising since the i_k 's and the v_k 's seem to bear so little relation to each other.

Tellegen's theorem Consider an arbitrary circuit. Let the digraph \mathcal{G} have b branches. Let us use associated reference directions. Let $\mathbf{i} = (i_1, i_2, \dots, i_b)^T$ be any set of branch currents satisfying KCL for \mathcal{G} and let $\mathbf{v} = (v_1, v_2, \dots, v_b)^T$ be any set of branch voltages satisfying KVL for \mathcal{G} , then

$$\sum_{k=1}^{b} v_k i_k = 0 \qquad \text{or equivalently} \qquad \mathbf{v}^T \mathbf{i} = 0 \tag{7.1}$$

PROOF For the connected digraph \mathcal{G}_{i}^{10} choose a datum node; hence its reduced matrix A is defined unambiguously. Since i satisfies KCL, we have

$$\mathbf{Ai} = \mathbf{0} \tag{7.2}$$

¹⁰ We again use a hinged graph to take care of graphs which are not connected.

Since v satisfies KVL and since we use associated reference directions, for some node-to-datum voltage vector e, we have

$$\mathbf{v} = \mathbf{A}^T \mathbf{e} \tag{7.3}$$

Using these two equations we obtain successively,

$$\mathbf{v}^T \mathbf{i} = (\mathbf{A}^T \mathbf{e})^T \mathbf{i} = \mathbf{e}^T (\mathbf{A}^T)^T \mathbf{i} = \mathbf{e}^T (\mathbf{A} \mathbf{i}) = 0$$
 (7.4)

where in the last step we used Eq. (7.2).

REMARKS

- 1. The v and the i in the theorem need not bear any relation to each other: v must *only* satisfy KVL and i must *only* satisfy KCL, and we must use associated reference directions.
- 2. Suppose that for the given connected digraph \mathcal{G} , let \mathbf{v}' and \mathbf{v}'' satisfy KVL, and let \mathbf{i}' and \mathbf{i}'' satisfy KCL. Then Tellegen's theorem asserts that

$$\mathbf{v}'^T \mathbf{i}' = 0$$
 $\mathbf{v}'^T \mathbf{i}'' = 0$ $\mathbf{v}''^T \mathbf{i}' = 0$ $\mathbf{v}''^T \mathbf{i}'' = 0$ (7.5)

Equation (7.5) is of particular interest. Note that \mathbf{v}' , \mathbf{v}'' , \mathbf{i}' , and \mathbf{i}'' are not related other than by the fact that they pertain to the same digraph and that they each independently satisfy Kirchhoff's laws. Clearly, Tellegen's theorem depicts only the interconnection properties of the circuit or the *topology* of the digraph. We will demonstrate later that this general form of Tellegen's theorem can be used to prove some general results in circuit theory.

7.2 Tellegen's Theorem and Conservation of Energy

Consider a lumped connected circuit and let us measure, at some time t, all its branch voltages $v_k(t)$ and all its branch currents $i_k(t)$, $k = 1, 2, \ldots, b$. Obviously $\mathbf{v}(t)$ and $\mathbf{i}(t)$ satisfy KVL and KCL, hence, by Tellegen's theorem

$$\mathbf{v}(t)^{T}\mathbf{i}(t) = \sum_{k=1}^{b} v_{k}(t) i_{k}(t) = 0$$
 (7.6)

Now, since we use associated reference directions, $v_k(t)i_k(t)$ is the power delivered, at time t, to branch k by the remainder of the circuit; equivalently, $v_k(t)i_k(t)$ is the rate at which energy is delivered, at time t, to branch k by the remainder of the circuit. Hence Eq. (7.6) asserts that the energy is conserved. Thus, for lumped circuits, conservation of energy is a consequence of Kirchhoff's laws.

To appreciate the fact that Tellegen's theorem is far more general than conservation of energy, work out the following exercise:

Exercise Consider an arbitrary circuit with digraph \mathcal{G} . Suppose that, for all $t \ge 0$, $\mathbf{v}(t)$ satisfies KVL for \mathcal{G} and $\mathbf{i}(t)$ satisfies KCL for \mathcal{G} . Show that for all t_1 , $t_2 \ge 0$

$$\sum_{k=1}^{b} v_k(t_1) i_k(t_2) = 0 \qquad \sum_{k=1}^{b} v_k(t_2) i_k(t_1) = 0$$
 (7.7a)

$$\sum_{k=1}^{b} v_k(t_1) \dot{i}_k(t_2) = 0 \qquad \sum_{k=1}^{b} \dot{v}_k(t_1) i_k(t_2) = 0$$
 (7.7b)

$$\sum_{k=1}^{b} v_k(t_2) \dot{i}_k(t_1) = 0 \qquad \sum_{k=1}^{b} \dot{v}_k(t_2) i_k(t_1) = 0$$
 (7.7c)

where $\dot{v}_k(t)$ denotes $dv_k/dt(t)$ and $\dot{i}_k(t)$ denotes $di_k/dt(t)$.

7.3 The Relation between Kirchhoff's Laws and Tellegen's Theorem

In circuit theory there are two fundamental postulates: KCL and KVL. We have proved that KCL and KVL imply Tellegen's theorem. It is interesting to note that any one of Kirchhoff's laws together with Tellegen's theorem implies the other. More precisely we have the following properties:

Properties

- 1. If, for all v satisfying KVL, $\mathbf{v}^T \mathbf{i} = 0$ then i satisfies KCL.
- 2. If, for all i satisfying KCL, $\mathbf{v}^T \mathbf{i} = 0$, then \mathbf{v} satisfies KVL.

PROOF

1. For all e let $v = A^T e$, and thus v satisfies KVL. By assumption,

$$0 = \mathbf{v}^T \mathbf{i} = \mathbf{e}^T \mathbf{A} \mathbf{i}$$

Now since e is an arbitrary node-to-datum voltage vector, the last equality implies Ai = 0, i.e., i satisfies KCL.

2. Let ℓ be an arbitrary loop in the graph \mathcal{G} . Consider the i obtained by assigning zero current to all branches of \mathcal{G} except for those of loop ℓ ; depending on whether the reference direction of branch j in loop ℓ agrees with that of loop ℓ , we assign i_j to be 1 A or -1 A. The resulting i satisfies KCL at all nodes of \mathcal{G} . Tellegen's theorem gives

$$\sum_{j=1}^{b} v_j i_j = \sum_{\substack{\text{over} \\ \text{branches} \\ \text{is loon } \ell}} \pm v_j = 0$$

thus the algebraic sum of the branch voltages around loop ℓ is zero, i.e., KVL holds for loop ℓ . Since ℓ is arbitrary, we have shown that KVL holds for all loops of \mathcal{G} .

7.4 Geometric Interpretation¹¹

In this section we shall use linear vector space to interpret the significance of Kirchhoff's laws and Tellegen's theorem. We will use the standard notations. For example, " \mathbb{R}^b " means "a b-dimensional vector space," " \in " means "is a member of," etc.

Tellegen's theorem requires that \mathbf{v} satisfy KVL and \mathbf{i} satisfy KCL for the given digraph \mathcal{G} . Let \mathcal{G} be connected and have b branches and n nodes. From Sec. 6.3, we have

KCL:
$$\mathbf{Ai} = \mathbf{0} \tag{7.8}$$

$$\mathbf{KVL}: \qquad \mathbf{v} = \mathbf{A}^T \mathbf{e} \tag{7.9}$$

We state the following properties based on the discussion of linear independence of equations.

KCL properties

- 1. The $(n-1) \times b$ matrix **A** is full rank, i.e., its n-1 rows are linearly independent vectors in the b-dimensional space \mathbb{R}^b . (7.10)
- 2. $Ai(t) = 0 \Leftrightarrow the b$ -dimensional current vector i(t) satisfies KCL. (7.11)
- 3. The set of all branch current vectors i that satisfy KCL form a subspace, called the KCL solution space, and we label it K_i . (7.12)
- 4. Since K_i is obtained by imposing n-1 linearly independent constraints on the *b*-dimensional current vector **i**, the *dimension* of K_i is b-n+1. (7.13)

The above implies

$$\begin{pmatrix} \mathbf{i} \in \mathbb{R}^n \\ \text{satisfies} \\ \text{KCL} \end{pmatrix} \Leftrightarrow (\mathbf{A}\mathbf{i} = \mathbf{0}) \Leftrightarrow (\mathbf{i} \in K_i)$$
 (7.14)

KVL properties

- 1. A^T has full column rank, i.e., its n-1 columns are linearly independent vectors in the b-dimensional space \mathbb{R}^b . (7.15)
- 2. For some (n-1)-dimensional vector $\mathbf{e}(t)$, $\mathbf{v}(t) = \mathbf{A}^T \mathbf{e}(t) \Leftrightarrow$ the b-dimensional vector $\mathbf{v}(t)$ satisfies KVL. (7.16)
- 3. The set of all v's satisfying KVL form a (n-1)-dimensional subspace which we call the KVL solution space K_v . (7.17)
- 4. Since the subspace K_n is spanned by n-1 linearly independent vectors, the dimension of K_n is n-1. (7.18)

¹¹ Advanced topic, may be omitted without loss of continuity.

The above implies

$$\begin{pmatrix} \mathbf{v} \in \mathbb{R}^{h} \\ \text{satisfies} \\ \mathbf{KVL} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \mathbf{v} = \mathbf{A}^{T} \mathbf{e} \\ \text{for some } \mathbf{e} \\ \text{in } \mathbb{R}^{n-1} \end{pmatrix} \Leftrightarrow (\mathbf{v} \in K_{v})$$
 (7.19)

Now Tellegen's theorem says that for any such $\mathbf{v} \in \mathbb{R}^b$ and any such $\mathbf{i} \in \mathbb{R}^b$, $\mathbf{v}^T \mathbf{i} = 0$, i.e., the vectors \mathbf{v} and \mathbf{i} are orthogonal.

So viewing the subspaces K_v and K_i as subspaces of the same vector space \mathbb{R}^b , Tellegen's theorem asserts that every vector in K_v is orthogonal to every vector of K_i . This is denoted by

$$K_{\nu} \perp K_{i} \tag{7.20}$$

i.e., the subspaces K_v and K_i are orthogonal. The orthogonality of K_v and K_i is illustrated in Fig. 7.2.

Recalling that the dimension of K_i is b-n+1 and that of K_v is n-1, the sum of their dimensions is b. Consequently the subspaces K_i and K_v are not only orthogonal, but also have their direct sum equal to \mathbb{R}^b . In other words, any vector in \mathbb{R}^b can be written uniquely as the sum of a vector in K_i and a vector in K_v .

To illustrate the equivalences in Eqs. (7.14) and (7.19) we consider two simple examples.

Example 1 \mathscr{G} is the digraph of a two-node three-branch circuit shown in Fig. 7.3; we see that **A** is a 1×3 matrix, namely,

So
$$A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

So $Ai = 0 \Leftrightarrow i_1 + i_2 + i_3 = 0$ (7.21)

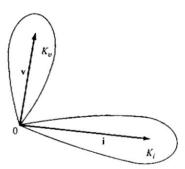


Figure 7.2 Figure illustrating the orthogonality of the subspaces K_i and K_v , where K_i is the set of all i's satisfying KCL and K_v is the set of all v's satisfying KVL.

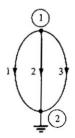


Figure 7.3 A digraph with two nodes and three branches.

$$\mathbf{v} = \mathbf{A}^T \mathbf{e} \Leftrightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e_1 \tag{7.22}$$

 K_i is a two-dimensional subspace; i_1 , i_2 , i_3 are constrained by one equation, the KCL at node ①, Eq. (7.21). K_i is shown in Fig. 7.4.

 K_{ν} is a one-dimensional subspace: There is only one degree of freedom, namely, the node voltage e_1 . [See Eq. (7.22).] K_{ν} is shown in Fig. 7.5. Note that the vector $(1, 1, 1)^T$ which spans K_{ν} is orthogonal to K_{ν} , as required by Tellegen's theorem.

Example 2 \mathscr{G} is the digraph of a three-node four-branch circuit shown in Fig. 7.6. Now A is a 2×4 matrix, namely

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix} \tag{7.23}$$

KCL, namely Ai = 0, reads

$$i_1 + i_3 + i_4 = 0 i_2 - i_3 - i_4 = 0$$
 (7.24)

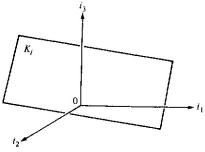


Figure 7.4 The two-dimensional KCL solution space of Example 1.

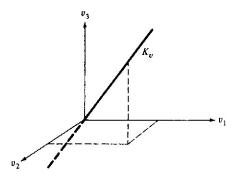


Figure 7.5 The one-dimensional KVL solution space of Example 1.

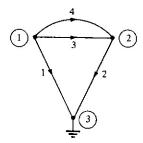


Figure 7.6 A digraph with three modes and four branches considered in Example 2.