\[ \begin{align*}
R_1C_1 = \tau_1 &= 0.15 \quad \omega_1 = 10 \text{rad/s} \\
R_2C_2 = \tau_2 &= 10^{-3} \quad \omega_2 = 10^3 \text{rad/s} \\
R_3C_3 = \tau_3 &= 10^{-9} \quad \omega_3 = 10^9 \text{rad/s} \\
\alpha &= 1000 \quad \alpha_2 = 1000
\end{align*} \]

\[\frac{\omega T \alpha}{\left(\frac{1}{\omega_2 T_2} \right) \left(\frac{1}{\omega_2 T_2} \right) \left(\frac{1}{\omega_3 - \omega_2} \right)}\]

\[A \quad B \quad C\]
\[
\frac{jw^2}{1+jwT_1} - \frac{1}{1+jwT_2} - \frac{a_2}{1+jwT_3}
\]

\[
\frac{w^2 T_1 T_3 - a_2 a_3}{(1+jwT_1)(1+jwT_2)(1+jwT_3)}
\]

\[
w < T_1 \quad |H| = w^2 T_1 T_3 a_2 a_3
\]

\[
|H|_{dB} = 20 \cdot \log_{10} (w^2) + 20 \cdot \log_{10} \left(\frac{T_1 T_3 a_2 a_3}{10^6}\right)
\]

\[
-10 \log_{10} (w)
\]
Lecture 6

- Last time:
  - Rapid sketching techniques for more complicated transfer functions

- Today:
  - 2nd order circuits in the time and frequency domains

A Second Order System

Where does the inductor come from?

Do step response: $v_S(t)$ jumps to $V_{DD}$ at $t = 0$
Step Response of L-R-C Circuit

Initial conditions: \( v_C(t=0) = 0 \text{ V}; i_L(t=0) = 0 \text{ A} \)

\[
i_L = i_C = \left( \frac{1}{L} \int_0^t v_L(t') \, dt' \right) = C \frac{dv_C}{dt}
\]

Inductor voltage: \( v_L = V_{DD} - \left( C \frac{dv_C}{dt} \right) R + v_C \)

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Solving the 2nd Order ODE

\[
LC \frac{d^2 v_C}{dt^2} + RC \frac{dv_C}{dt} + v_C = V_{DD}
\]

Steady-state solution: \( v_{C,ss} = V_{DD} \ (t \to \infty) \)

Transient solution: \( v_{C,tr} = ? \) … guess \( v_{C,tr} = ae^{bt} \)

and substitute: \( LC \frac{d^2 (ae^{bt})}{dt^2} + RC (ae^{bt}) + ae^{bt} = 0 \)

\[
s^2 + \left( \frac{1}{RC} \right) s + \left( \frac{1}{LC} \right) = 0
\]

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Characteristic Equation

\[ s^2 + \left( \frac{R}{2L} \right)s + \left( \frac{1}{LC} \right) = 0 \]

Use quadratic formula to find the roots:

\[ s_{1,2} = -\left( \frac{R}{2L} \right) \pm \sqrt{\left( \frac{R}{2L} \right)^2 - \left( \frac{1}{LC} \right)} \]

\[ V_{out}(t) = a e^{st} \]

Underdamped Case

\[ \left( \frac{R}{2L} \right)^2 - \frac{1}{LC} < 0 \]

\[ s_{1,2} = -\left( \frac{R}{2L} \right) \pm j\sqrt{\left( \frac{1}{LC} \right) - \left( \frac{R}{2L} \right)^2} \]

\[ V_C(t) = V_{DD} + a_1 e^{-(R/2L)t} e^{j\sqrt{(1/LC) - (R/2L)^2}t} + a_2 e^{-(R/2L)t} e^{-j\sqrt{(1/LC) - (R/2L)^2}t} \]

Form of solution...

\[ V_C(t) = V_{DD} + \sum \alpha e^{\omega_d t} \cos(\omega t) \]

\[ -at \]
Qualitative Underdamped Waveform

Extreme Underdamped Case

Exponential decay time is set by $\alpha = \frac{R}{2L}$

Small $\frac{R}{L} \rightarrow$ decay takes a long time and oscillation has a frequency that's nearly $\sqrt{\frac{1}{LC}}$

Number of cycles during “ringdown” is

$$N \approx \frac{\frac{1}{\alpha}}{\frac{1}{\sqrt{\frac{1}{LC}}}} = \frac{2L}{R} \frac{2}{\sqrt{LC}}$$

What happens when $R = 0 \Omega$?
thin-Film Bulk Acoustic Resonator (FBAR)

- Agilent Technologies
  *IEEE ISSCC 2001*
  2 GHz resonator
- $N > 1000$
- Brian Otis, Jan Rabaey
  (BWRC): low-noise oscillator
- Equivalent Circuit:

Phasor Analysis of 2\textsuperscript{nd} Order Circuit

Impedance divider:

$$V_c = V_s \left[ \frac{1/j\omega C}{1/j\omega C + R + j\omega L} \right]$$
Transfer Function

Simplifying:

\[ H(j\omega) = \frac{1}{1 + j\omega RC - \omega^2 LC} \]

Define parameters: \( \omega_o = 1/\sqrt{LC} \)
\( \tau = RC \)

\[ H(j\omega) = \frac{1}{1 - (\omega/\omega_o)^2 + j\omega\tau} \]

Limiting Cases: Magnitude and Phase

Low frequency: \( \omega << \omega_o \) \( |H| = 1 \) \( \angle H = 0^\circ \)

High frequency: \( \omega >> \omega_o \) \( |H| = \frac{1}{(\omega/\omega_o)^2} \) \( \angle H = 180^\circ \)

Resonant frequency: \( \omega = \omega_o \) \( |H| = \frac{1}{\omega_o} \) \( \angle H = -90^\circ \)

\[ \frac{1}{\omega_0\tau} = \frac{1}{\sqrt{\frac{V_{cc}}{RC}}} = \frac{\sqrt{V_{cc}}}{RC} = \frac{i}{i} \sqrt{\frac{E}{C}} \]
Inductor-Capacitor "Tuning"

At resonance, the impedance of the capacitor cancels
the impedance of the inductor $\rightarrow$ phasor current is
maximum and capacitor voltage peaks

How "sharp" or "narrow" is the resonance?

Define the quality factor $Q = \frac{\omega_0}{\Delta \omega} \rightarrow Q = \frac{1}{\omega_0 \tau}$

Magnitude Bode Plot