LTI: Linear Time-Invariant System

• System is linear (studied thoroughly in 16AB):

• System is time invariant:
  – There is no “clock” or time reference
  – The transfer function is not a function of time
  – It does not matter when you apply the input. The transfer function is going to be the same …
Linear Systems

• Continuous time linear systems have a lot in common with finite dimensional linear systems we studied in 16AB:
  – Linearity:
  – Basis Vectors $\rightarrow$ basis functions:
  – Superposition:
  – Matrix Representation $\rightarrow$ Integral representation:

Linear Systems (cont)

• Eigenvectors $\rightarrow$ eigenfunctions

• Orthonormal basis

• Eigenfunction expansion

• Operators acting on eigenfunction expansion
LTI Systems

- Since most periodic (non-periodic) signals can be decomposed into a summation (integration) of sinusoids via Fourier Series (Transform), the response of a LTI system to virtually any input is characterized by the frequency response of the system:

Example: Low Pass Filter (LPF)

- Input signal: \( v_s(t) = V_s \cos(\omega t) \)
- We know that:
  \[
  v_o(t) = K \cdot V_s \cos(\omega t + \phi)
  \]

\[
\begin{align*}
  v_b(t) &= v_s(t) - i(t)R \\
i(t) &= C \frac{dv_b}{dt} \\
v_b(t) &= v_s(t) - RC \frac{dv_b}{dt} \\
v_s(t) &= v_b(t) + \frac{dv_b}{dt}
\end{align*}
\]
Complex Exponential

- Euler’s Theorem says that
  \[ e^{ix} = \cos x + j \sin x \]
- This can be derived by expanding each term in a power series.
- If take the magnitude of this quantity, it’s unity
  \[ |e^{ix}| = \sqrt{\cos^2 x + \sin^2 x} = 1 \]
- That means that \( e^{i\phi} \) is a point on the unit circle at an angle of \( \phi \) from the x-axis.

Any complex number \( z \), expressed as a real and imaginary part \( z = x + jy \), can also be interpreted as having a magnitude and a phase. The magnitude \( |z| = \sqrt{x^2 + y^2} \) and the phase \( \phi = \angle z = \tan^{-1} \frac{y}{x} \) can be combined using the complex exponential

\[ x + jy = |z|e^{j\phi} \]

The Rotating Complex Exponential

- So the complex exponential is nothing but a point tracing out a unit circle on the complex plane:

\[ e^{ix} = \cos x + i \sin x \]
Magic: Turn Diff Eq into Algebraic Eq

- Integration and differentiation are trivial with complex numbers:
  \[
  \frac{d}{dt} e^{i\omega t} = i\omega e^{i\omega t} \quad \int e^{i\omega t} d\tau = \frac{1}{i\omega} e^{i\omega t}
  \]

- Any ODE is now trivial algebraic manipulations … in fact, we’ll show that you don’t even need to directly derive the ODE by using phasors

- The key is to observe that the current/voltage relation for any element can be derived for complex exponential excitation

Complex Exponential is Powerful

- To find steady state response we can excite the system with a complex exponential

\[
\begin{align*}
& e^{i\omega t} \rightarrow \text{LTI System} \\
& H \rightarrow |H(\omega)| e^{i(\alpha t + \phi)} \rightarrow \text{Mag Response} \quad \text{Phase Response}
\end{align*}
\]

- At any frequency, the system response is characterized by a single complex number H:
  \[
  |H(\omega)| \quad \phi = \angle H(\omega)
  \]

- This is not surprising since a sinusoid is a sum of complex exponentials (and because of linearity!)

\[
\begin{align*}
\sin(\omega t) &= \frac{e^{i\omega t} - e^{-i\omega t}}{2i} \\
\cos(\omega t) &= \frac{e^{i\omega t} + e^{-i\omega t}}{2}
\end{align*}
\]

- From this perspective, the complex exponential is even more fundamental
Solving LPF with Phasors

• Let’s excite the system with a complex exp:

\[
v_s(t) = v_0(t) + \tau \frac{dv_0}{dt}
\]

\[
v_s(t) = V_s e^{j\omega t}
\]

\[
v_o(t) = V_o e^{j(\omega t + \phi)} = V_o e^{j\omega t}
\]

use \(j\) to avoid confusion

real complex

\[
V_s e^{j\phi} = V_0 e^{j\phi} + \tau \cdot j\omega \cdot V_0 e^{j\phi}
\]

\[
V_s = V_0 \left(1 + j\omega \cdot \tau\right)
\]

\[
\frac{V_o}{V_s} = \frac{1}{\left(1 + j\omega \cdot \tau\right)}
\]

Easy!!

Magnitude and Phase Response

• The system is characterized by the complex function

\[
H(\omega) = \frac{V_o}{V_s} = \frac{1}{\left(1 + j\omega \cdot \tau\right)}
\]

• The magnitude and phase response match our previous calculation:

\[
|H(\omega)| = \left|\frac{V_o}{V_s}\right| = \frac{1}{\sqrt{1 + (\omega \tau)^2}}
\]

\[
\angle H(\omega) = -\tan^{-1} \omega \tau
\]
Why did it work?

- Again, the system is linear:
  \[ y = L(x_1 + x_2) = L(x_1) + L(x_2) \]

- To find the response to a sinusoid, we can find the response to \( e^{i\omega t} \) and \( e^{-i\omega t} \) and sum the results:

\[
\begin{align*}
  e^{i\omega t} &\rightarrow \text{LTI System } H &  \rightarrow |H(\omega)|e^{i(\omega t + \phi_1)} \\
  e^{-i\omega t} &\rightarrow \text{LTI System } H &  \rightarrow |H(-\omega)|e^{i(-\omega t + \phi_2)} \\
  \frac{e^{i\omega t} + e^{-i\omega t}}{2} &\rightarrow \text{LTI System } H &  \rightarrow \frac{H(\omega)e^{i\omega t} + H(-\omega)e^{-i\omega t}}{2}
\end{align*}
\]

(cont.)

- Since the input is real, the output has to be real:

\[
y(t) = \frac{H(\omega)e^{i\omega t} + H(-\omega)e^{-i\omega t}}{2}
\]

- That means the second term is the conjugate of the first:

\[
|H(-\omega)| = |H(\omega)| \quad \text{(even function)}
\]

\[
\angle H(-\omega) = -\angle H(\omega) = -\phi \quad \text{(odd function)}
\]

- Therefore the output is:

\[
y(t) = \frac{|H(\omega)|}{2} \left( e^{i(\omega t + \phi)} + e^{-i(\omega t + \phi)} \right)
\]

\[
= |H(\omega)|\cos(\omega t + \phi)
\]
“Proof” for Linear Systems

- For an arbitrary linear circuit (L,C,R,M, and dependent sources), decompose it into linear sub-operators, like multiplication by constants, time derivatives, or integrals:

\[ y = L(x) = ax + b_1 \frac{d}{dt} x + b_2 \frac{d^2}{dt^2} x + \cdots + \int x + \int \int \cdots \int x + \cdots \]

- For a complex exponential input \( x \) this simplifies to:

\[
y = L(e^{j\omega t}) = ae^{j\omega t} + b_1 \frac{d}{dt} e^{j\omega t} + b_2 \frac{d^2}{dt^2} e^{j\omega t} + \cdots + c_1 \int e^{j\omega t} + c_2 \int \int e^{j\omega t} + \cdots
\]

\[
y = ae^{j\omega t} + b_1 j\omega e^{j\omega t} + b_2 (j\omega)^2 e^{j\omega t} + \cdots + c_1 \frac{e^{j\omega t}}{j\omega} + c_2 \frac{e^{j\omega t}}{(j\omega)^2} + \cdots
\]

\[
y = Hx = e^{j\omega t} \left( a + b_1 j\omega + b_2 (j\omega)^2 + \cdots + \frac{c_1}{j\omega} + \frac{c_2}{(j\omega)^2} + \cdots \right)
\]

“Proof” (cont.)

- Notice that the output is also a complex exp times a complex number:

\[
y = Hx = e^{j\omega t} \left( a + b_1 j\omega + b_2 (j\omega)^2 + \cdots + \frac{c_1}{j\omega} + \frac{c_2}{(j\omega)^2} + \cdots \right)
\]

- The amplitude of the output is the magnitude of the complex number and the phase of the output is the phase of the complex number

\[
y = Hx = e^{j\omega t} \left| H(\omega) \right| e^{jH(\omega)}
\]

\[
\text{Re}[y] = \left| H(\omega) \right| \cos(\omega t + \angle H(\omega))
\]
Phasors

- With our new confidence in complex numbers, we go full steam ahead and work directly with them … we can even drop the time factor $e^{i\omega t}$ since it will cancel out of the equations.
- Excite system with a phasor: $\vec{V}_1 = V_1 e^{i\phi_1}$
- Response will also be phasor: $\vec{V}_2 = V_2 e^{i\phi_2}$
- For those with a Linear System background, we’re going to work in the frequency domain
  - This is the Laplace domain with $s = j\omega$

Capacitor I-V Phasor Relation

- Find the Phasor relation for current and voltage in a cap:

$$i_c(t) = C \frac{dv_c(t)}{dt} \quad v_c(t) = V_c e^{j\omega t} \quad v_c(t) = V_c e^{j\omega t}$$

$$i_c(t) = I_c e^{j\omega t}$$

$$I_c e^{j\omega t} = C \frac{d}{dt} [V_c e^{j\omega t}]$$

$$CV_c \frac{d}{dt} e^{j\omega t} = j\omega CV_c e^{j\omega t}$$

$$I_c e^{j\omega t} = j\omega CV_c e^{j\omega t}$$

$$I_c = j\omega CV_c$$
Inductor I-V Phasor Relation

- Find the Phasor relation for current and voltage in an inductor:

\[ v(t) = L \frac{di(t)}{dt} \]
\[ i(t) = I e^{j\omega t} \]
\[ v(t) = V e^{j\omega t} \]
\[ V e^{j\omega t} = L \frac{d}{dt} [I e^{j\omega t}] \]
\[ LI \frac{d}{dt} e^{j\omega t} = j\omega LL e^{j\omega t} \]
\[ V e^{j\omega t} = j\omega LL e^{j\omega t} \]
\[ V = j\omega LI \]

Impede the Currents!

- Suppose that the “input” is defined as the voltage of a terminal pair (port) and the “output” is defined as the current into the port:

\[ v(t) = V e^{j\omega t} = |V| e^{j(\omega t + \phi_v)} \]
\[ i(t) = I e^{j\omega t} = |I| e^{j(\omega t + \phi_i)} \]

- The impedance \( Z \) is defined as the ratio of the phasor voltage to phasor current (“self” transfer function)

\[ Z(\omega) = H(\omega) = \frac{V}{I} = \left| \frac{V}{I} \right| e^{j(\phi_v - \phi_i)} \]
Admit the Currents!

- Suppose that the “input” is defined as the current of a terminal pair (port) and the “output” is defined as the voltage into the port:

\[ v(t) = V e^{j\omega t} = |V| e^{j(\omega t + \phi_v)} \]
\[ i(t) = I e^{j\omega t} = |I| e^{j(\omega t + \phi_i)} \]

- The admittance \( Y \) is defined as the ratio of the phasor current to phasor voltage (“self” transfer function)

\[ Y(\omega) = H(\omega) = \frac{I}{V} = \frac{I}{V} e^{j(\omega t - \phi)} \]

Voltage and Current Gain

- The voltage (current) gain is just the voltage (current) transfer function from one port to another port:

\[ G_v(\omega) = \frac{V_2}{V_1} = \frac{V_2}{V_1} e^{j(\omega t - \phi_v)} \]
\[ G_i(\omega) = \frac{I_2}{I_1} = \frac{I_2}{I_1} e^{j(\omega t - \phi_i)} \]

- If \(|G| > 1\), the circuit has voltage (current) gain
- If \(|G| < 1\), the circuit has loss or attenuation
Transimpedance/admittance

- Current/voltage gain are unit-less quantities
- Sometimes we are interested in the transfer of voltage to current or vice versa

\[ J(\omega) = \frac{V_2}{I_1} = \frac{V_2}{I_1} e^{j(\phi_2 - \phi_1)} \quad [\Omega] \]

\[ K(\omega) = \frac{I_2}{V_1} = \left| \frac{I_2}{V_1} \right| e^{j(\phi_2 - \phi_1)} \quad [S] \]

Direct Calculation of H (no DEs)

- To directly calculate the transfer function (impedance, trans-impedance, etc) we can generalize the circuit analysis concept from the “real” domain to the “phasor” domain
- With the concept of impedance (admittance), we can now directly analyze a circuit without explicitly writing down any differential equations
- Use KVL, KCL, mesh analysis, loop analysis, or node analysis where inductors and capacitors are treated as complex resistors
LPF Example: Again!

- Instead of setting up the DE in the time-domain, let's do it directly in the frequency domain.
- Treat the capacitor as an imaginary “resistance” or impedance:

  \[
  Z_R = R \\
  Z_C = \frac{1}{j\omega C}
  \]

Bode Plots

- Simply the log-log plot of the magnitude and phase response of a circuit (impedance, transimpedance, gain, ...)
- Gives insight into the behavior of a circuit as a function of frequency.
- The “log” expands the scale so that breakpoints in the transfer function are clearly delineated.
- In EECS 140, Bode plots are used to “compensate” circuits in feedback loops.
**Frequency Response of Low-Pass Filters**

\[ T(\omega) = \frac{1}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\omega / \omega_0} \]

\[ \omega_0 = \frac{1}{RC} \]

\[ |T(\omega)| = \frac{1}{\sqrt{1 + (\omega / \omega_0)^2}} \]

\[ \angle T(\omega) = -\tan^{-1}(\omega / \omega_0) \]

\[ \omega_{0db} = \omega_0 \] [rad/sec]

\[ f_{3db} = \frac{\omega_0}{2\pi} \] [Hz]

**Frequency Response of High-Pass Filters**

\[ T(\omega) = \frac{R}{R + \frac{1}{j\omega C}} = \frac{1}{1 - j\omega RC} = \frac{1}{1 - j\omega / \omega_0} \]

\[ \omega_0 = \frac{1}{RC} \]

\[ |T(\omega)| = \frac{1}{\sqrt{1 + (\omega_0 / \omega)^2}} \]

\[ \angle T(\omega) = \tan^{-1}(\omega_0 / \omega) \]

\[ \omega_{0db} = \omega_0 \] [rad/sec]

\[ f_{3db} = \frac{\omega_0}{2\pi} \] [Hz]
Example: High-Pass Filter

• Using the voltage divider rule:

\[
H(\omega) = \frac{j\omega L}{R + j\omega L} = \frac{j\omega L}{1 + j\omega \frac{L}{R}}
\]

\[
H(\omega) = \frac{j\omega \tau}{1 + j\omega \tau}
\]

\[
\omega \to \infty \quad |H| \to \left| \frac{j\omega \tau}{1 + j\omega \tau} \right| = 1
\]

\[
\omega \to 0 \quad |H| \to \frac{0}{1 + 0} = 0
\]

\[
\omega = \frac{1}{\tau} \quad |H| = \left| \frac{j}{1 + j} \right| = \frac{1}{\sqrt{2}}
\]

HPF Magnitude Bode Plot

• Recall that log of product is the sum of log

\[
|H(\omega)|_{\text{dB}} = |j\omega \tau|_{\text{dB}} = |j\omega \tau|_{\text{dB}} + \left| \frac{1}{1 + j\omega \tau} \right|_{\text{dB}}
\]

\[
|j\omega \tau|_{\text{dB}} \quad \text{Increase by 20 dB/decade}
\]

\[
\omega \tau = 1 \Rightarrow |j\omega \tau|_{\text{dB}} = 0 \text{dB}
\]

Equals unity at breakpoint
**HPF Bode Plot (dissection)**

- The second term can be further dissected:

\[
\frac{1}{1 + j\omega\tau} = 0 \text{ dB} - |1 + j\omega\tau|_{\text{dB}}
\]

- Composite is simply the sum of each component:

\[
|j\omega\tau|_{\text{dB}} + \frac{1}{1 + j\omega\tau} = 0 \text{ dB} - |1 + j\omega\tau|_{\text{dB}}
\]

**Composite Plot**

- Composite is simply the sum of each component:

\[
|j\omega\tau|_{\text{dB}} + \frac{1}{1 + j\omega\tau} = 0 \text{ dB} - |1 + j\omega\tau|_{\text{dB}}
\]
Approximate versus Actual Plot

- Approximate curve accurate away from breakpoint
- At breakpoint there is a 3 dB error

HPF Phase Plot

- Phase can be naturally decomposed as well:
  \[
  H(\omega) = \frac{j\omega \tau}{1 + j\omega \tau} = j\omega \tau + \frac{1}{1 + j\omega \tau} = \frac{\pi}{2} - \tan^{-1}\omega \tau
  \]

- First term is simply a constant phase of 90 degrees
- The second term is the arctan function
- Estimate arctan function:
Power Flow

- The instantaneous power flow into any element is the product of the voltage and current: \( P(t) = i(t)v(t) \)

- For a periodic excitation, the average power is:
  \[
P_{av} = \int_T i(\tau)v(\tau)d\tau
  \]

- In terms of sinusoids we have
  \[
P_{av} = \int_T |I| |V| \cos(\omega t + \varphi_i) \cos(\omega t + \varphi_v) d\tau
  \]
  \[
  = |I| |V| \int_0^T \cos(\omega t + \varphi_i) \cos(\omega t + \varphi_v) d\tau
  \]
  \[
  = |I| |V| \int_0^T \cos(\omega t + \varphi_i) \cos(\omega t + \varphi_v) d\tau
  \]
  \[
  = \frac{|I| |V|}{2} \cos(\varphi_i - \varphi_v)
  \]

Power Flow with Phasors

\[
P_{av} = \frac{|I| |V|}{2} \cos(\varphi_i - \varphi_v)
\]

- Note that if \( \varphi_i - \varphi_v = \frac{\pi}{2} \), then \( P_{av} = \frac{|I| |V|}{2} \cos \left( \frac{\pi}{2} \right) = 0 \)

- Important: Power is a non-linear function so we can’t simply take the real part of the product of the phasors:
  \( P \neq \operatorname{Re}[I \cdot V] \)

- From our previous calculation:
  \[
P = \frac{|I| |V|}{2} \cos(\varphi_i - \varphi_v) = \frac{1}{2} \operatorname{Re}[I \cdot V^*] = \frac{1}{2} \operatorname{Re}[I^* \cdot V]
  \]
More Power to You!

• In terms of the circuit impedance we have:

\[
P = \frac{1}{2} \text{Re}[I \cdot V^*] = \frac{1}{2} \text{Re}\left[\frac{V}{Z} \cdot V^*\right] = \frac{|V|^2}{2} \text{Re}[Z^{-1}]
\]

\[
= \frac{|V|^2}{2} \text{Re} \left[\frac{Z^*}{|Z|^2}\right] = \frac{|V|^2}{2|Z|^2} \text{Re}[Z^*] = \frac{|V|^2}{2|Z|^2} \text{Re}[Z]
\]

• Check the result for a real impedance (resistor)

• Also, in terms of current:

\[
P = \frac{1}{2} \text{Re}[I^* \cdot V] = \frac{1}{2} \text{Re}[I^* \cdot I \cdot Z] = \frac{|I|^2}{2} \text{Re}[Z]
\]

Summary

• Complex exponentials are eigen-functions of LTI systems
  – Steady-state response of LCR circuits are LTI systems
  – Phasor analysis allows us to treat all LCR circuits as simple “resistive” circuits by using the concept of impedance (admittance)

• Frequency response allows us to completely characterize a system
  – Any input can be decomposed into either a continuum or discrete sum of frequency components
  – The transfer function is usually plotted in the log-log domain (Bode plot) – magnitude and phase
  – Location of poles/zeros is key