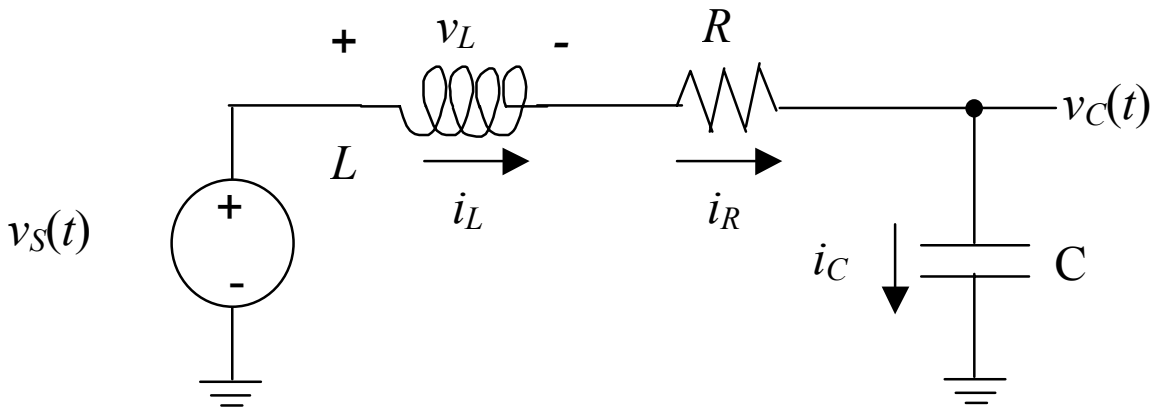
Lecture 6

- Last time:
  - Rapid sketching techniques for more complicated transfer functions
- Today :
  - 2^{nd} order circuits in the time and frequency domains

A Second Order System



Where does the inductor come from?

Do step response: $v_S(t)$ jumps to V_{DD} at $t = 0$

Step Response of L-R-C Circuit

Initial conditions: $v_C(t{=}0) = 0$ V; $i_L\,(t{=}0) = 0$ A

$$i_L = i_C \longrightarrow \left(\frac{1}{L}\right)\int_0^t v_L(t')dt' = C\frac{dv_C}{dt}$$

Inductor voltage: $v_L = V_{DD} - \left(\left(C\frac{dv_C}{dt}\right)R + v_C\right)$

Solving the 2nd Order ODE

$$LC\frac{d^2v_C}{dt^2}+RC\frac{dv_C}{dt}+v_C=V_{DD}$$

Steady-state solution: $v_{C,ss}=V_{DD}\ \ (t\to\infty)$

Transient solution: $v_{C,tr}=?$ … guess $v_{C,tr}=ae^{st}$

and substitute: $LCs^2\left(ae^{st}\right)+RCs\left(ae^{st}\right)+ae^{st}=0$

$$s^2+\left(L/R\right)s+\left(1/(LC)\right)=0$$

Characteristic Equation

$$s^2 + (L/R)s + (1/(LC)) = 0$$

Use quadratic formula to find the roots:

$$s_{1,2} = -\left(\frac{L}{2R}\right) \pm \sqrt{\left(\frac{L}{2R}\right)^2 - \left(\frac{1}{LC}\right)}$$

Underdamped Case

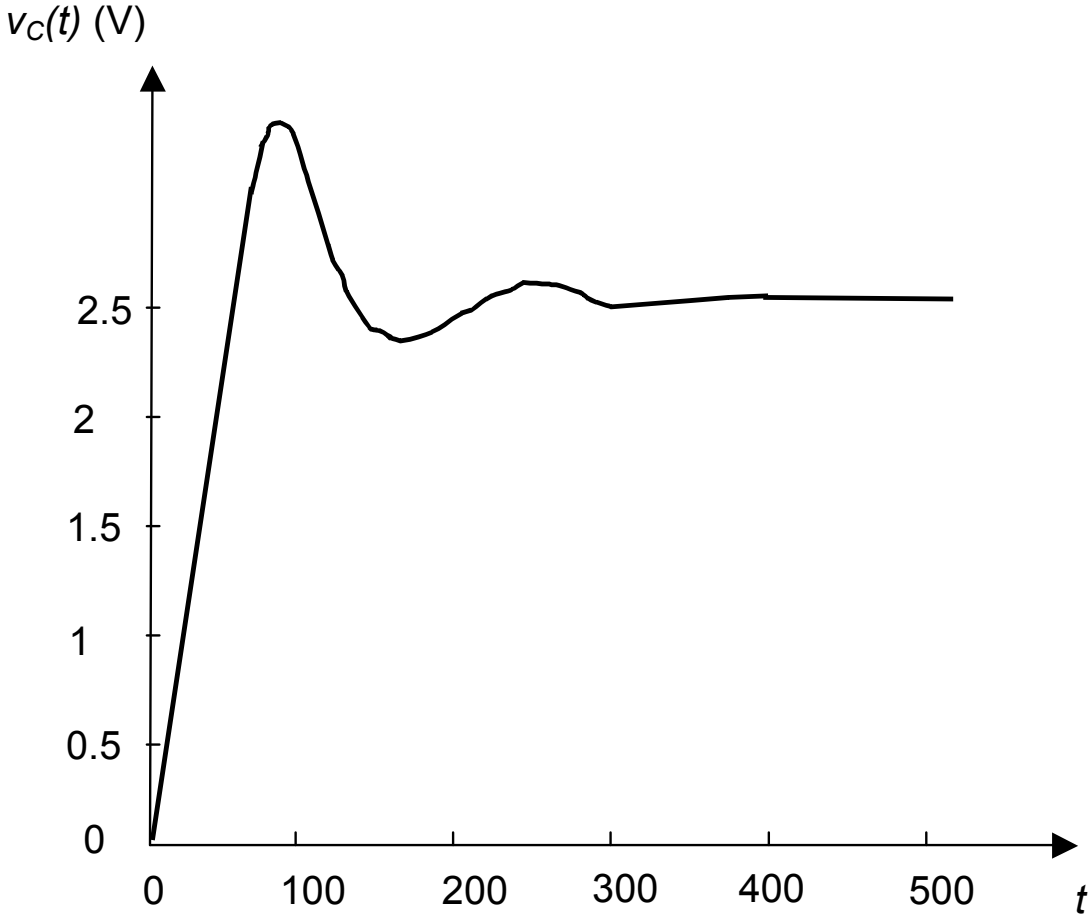
$$s_{1,2} = -\left(\frac{R}{2L}\right) \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right)} = -\left(\frac{R}{2L}\right) \pm j\sqrt{\left(\frac{1}{LC}\right) - \left(\frac{R}{2L}\right)^2}$$

$$v_C(t) = V_{DD} + a_1 e^{-(R/2L)t} e^{j\sqrt{(1/LC)-(R/2L)^2}\,t} +$$

$$a_2 e^{-\underbrace{(R/2L)}_{\alpha}t} e^{-j\underbrace{\sqrt{(1/LC)-(R/2L)^2}}_{\omega_d}t}$$

Form of solution …

Qualitative Underdamped Waveform



Extreme Underdamped Case

Exponential decay time is set by $\alpha = R/(2L)$

Small *R/L* → decay takes a long time and oscillation has a frequency that's nearly $\sqrt{1/(LC)}$

Number of cycles during "ringdown" is

$$N \approx \frac{(1/\alpha)}{(1/\sqrt{1/(LC)})} = \frac{2L/R}{\sqrt{LC}} = \frac{2}{R}\sqrt{\frac{L}{C}}$$

What happens when $R = 0\ \Omega$?

thin-Film Bulk Acoustic Resonator (FBAR)

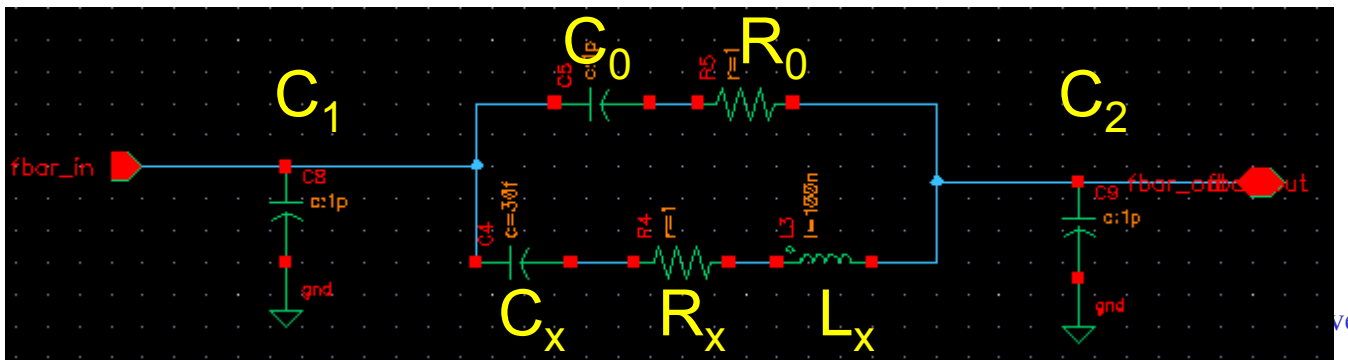
- Agilent Technologies
  IEEE ISSCC 2001

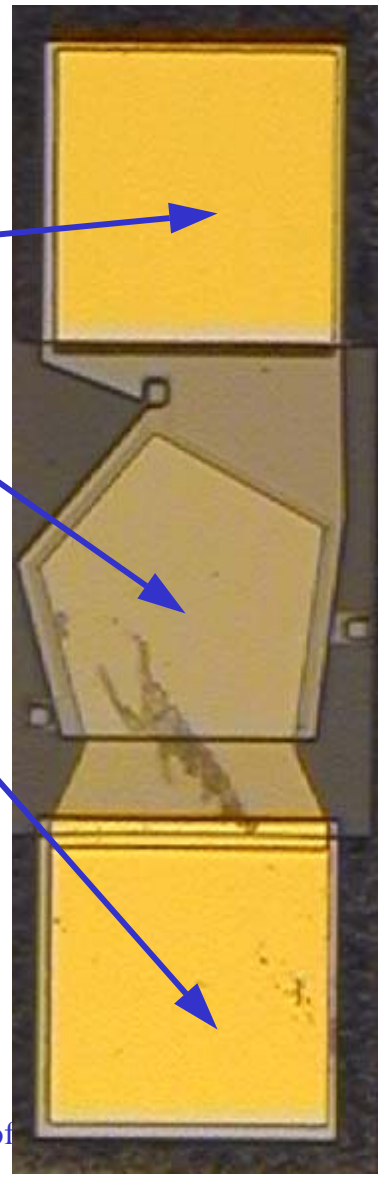
  2 GHz resonator

- *N* > 1000

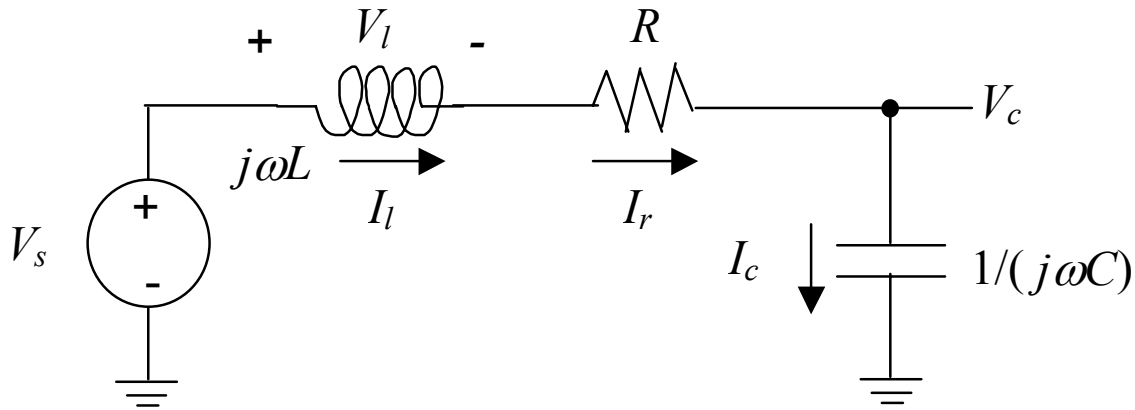
•Brian Otis, Jan Rabaey (BWRC): low-noise oscillator

•Equivalent Circuit:





Phasor Analysis of 2nd Order Circuit



Impedance divider:

$$V_c = V_s\left[\frac{1/j\omega C}{(1/j\omega C)+R+(j\omega L)}\right]$$

Transfer Function

Simplifying:

$$H(j\omega)=\left[\frac{1}{1+j\omega RC-\omega^2 LC)}\right]$$

Define parameters: $\omega_o = 1/\sqrt{LC}$ $\tau = RC$

$$H(j\omega)=\left[\frac{1}{1-(\omega/\omega_o)^2+j\omega\tau)}\right]$$

Limiting Cases: Magnitude and Phase

Low frequency: $\omega << \omega_o$

High frequency: $\omega >> \omega_o$

Resonant frequency: $\omega = \omega_o$

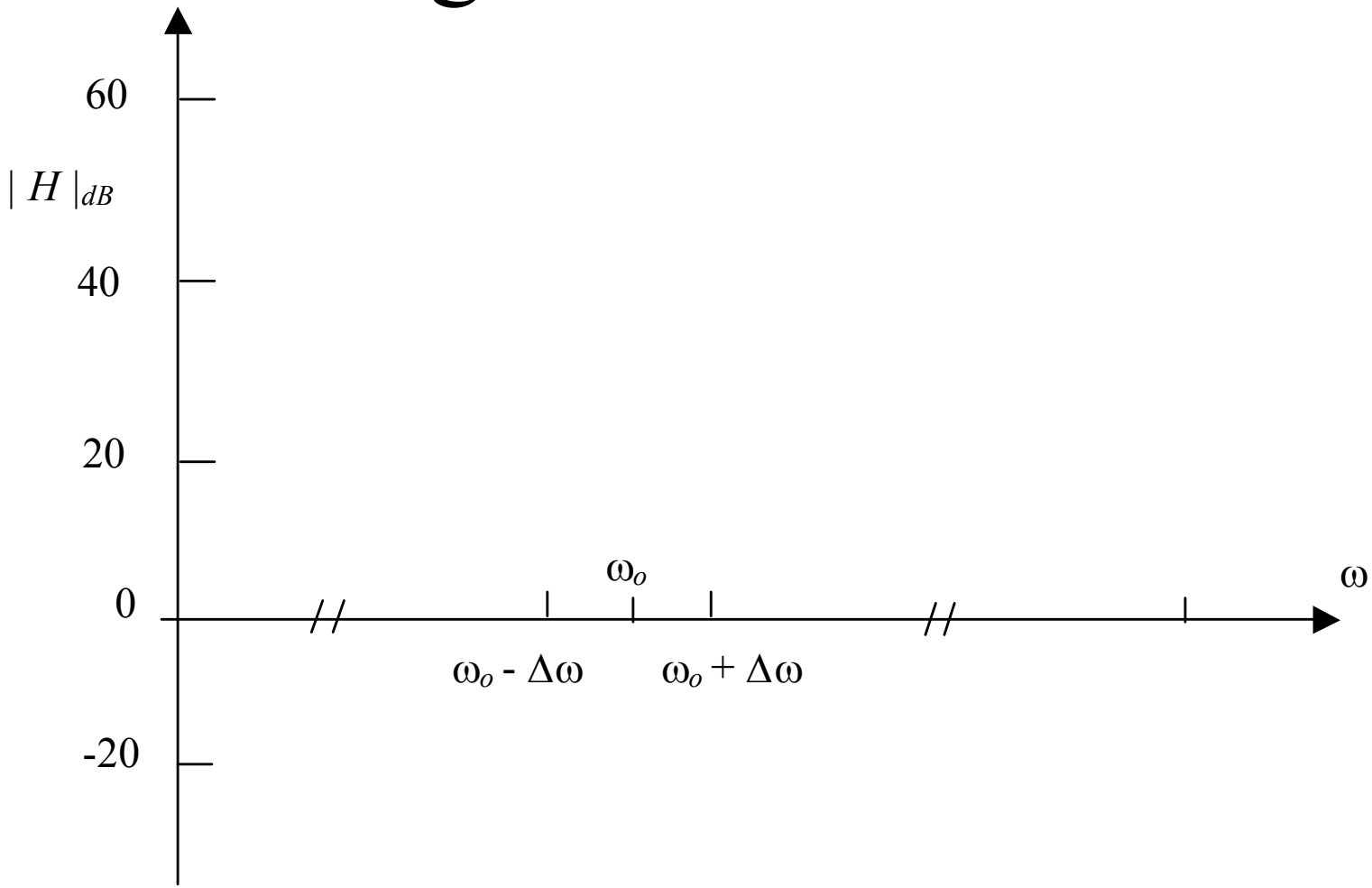
Inductor-Capacitor "Tuning"

At resonance, the impedance of the capacitor cancels the impedance of the inductor → phasor current is maximum and capacitor voltage peaks

How "sharp" or "narrow" is the resonance?

Define the quality factor $Q = \frac{\omega_o}{\Delta\omega} \longrightarrow Q = \frac{1}{\omega_o \tau}$

Magnitude Bode Plot



Phase Bode Plot