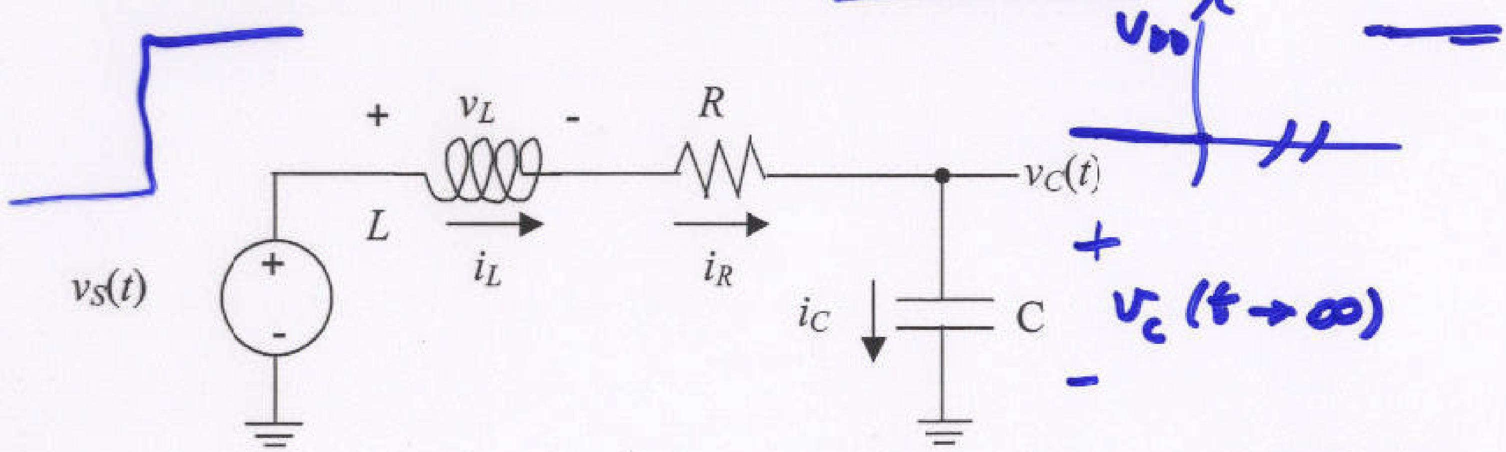
Lecture 6

- Last time:
  - Rapid sketching techniques for more complicated transfer functions
- Today : –RLC–
  - 2nd order circuits in the time and frequency domains

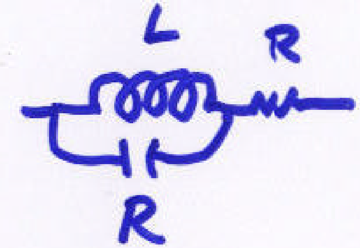
"FIRST ORDER"

A Second Order System

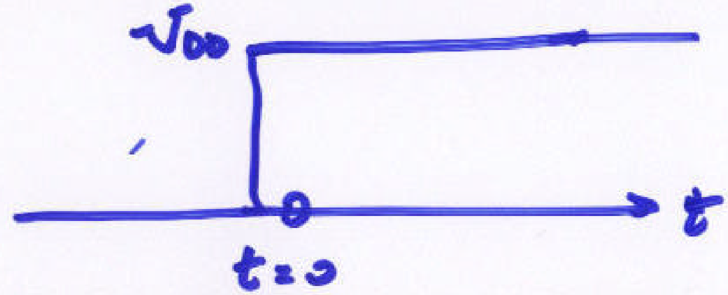


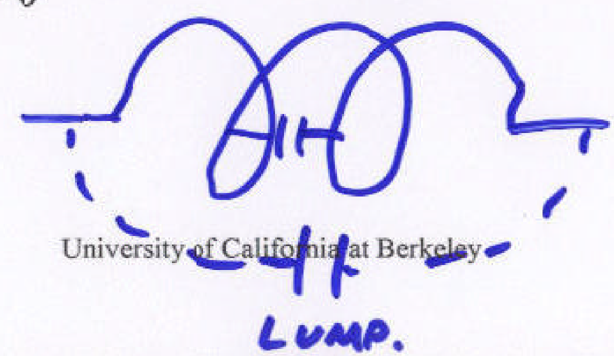
10x ↓ $j\omega L$.

- Where does the inductor come from? MAKE ONE!



→ Do step response: $v_S(t)$ jumps to V_{DD} at $t = 0$





Step Response of L-R-C Circuit

Initial conditions: $v_C(t=0) = 0\text{ V};\ i_L\,(t=0) = 0\text{ A}$

$i_L(t)$

$$i_L = i_C \longrightarrow \left(\frac{1}{L}\right)\int_0^t v_L(t')dt' = C\frac{dv_C}{dt}$$

Inductor voltage: $v_L = V_{DD} - \left(\left(C\frac{dv_C}{dt}\right)R + v_C\right)$

Solving the 2nd Order ODE

$$\boxed{LC\frac{d^2v_C}{dt^2}+RC\frac{dv_C}{dt}+v_C=V_{DD}}$$

- Steady-state solution: $v_{C,ss}=V_{DD}\ (t\to\infty)$

complex.

Transient solution: $v_{C,tr}=?$ … guess $v_{C,tr}=ae^{st}$

and substitute: $LCs^2\left(\cancel{ae^{st}}\right)+RCs\left(\cancel{ae^{st}}\right)+\cancel{ae^{st}}=0$

$$\boxed{s^2+(L/R)s+(1/(LC))=0}$$

Characteristic Equation

$s = ?$

$$\rightarrow s^2 + (L/R)s + (1/(LC)) = 0$$

Use quadratic formula to find the roots:

OVER DAMPED

$$s_{1,2} = -\left(\frac{R}{2L}\right) \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right)}$$

CRITICALLY DAMPED

e^{st}

IDENTICAL ROOTS →

REAL ROOTS

BORING!

"UNDER DAMPED"

→ $S_{1,2}$ COMPLEX

Underdamped Case

$$s_{1,2} = -\underbrace{\left(\frac{R}{2L}\right)}_{\text{REAL}} \pm \sqrt{\underbrace{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right)}_{<0}} = -\left(\frac{R}{2L}\right) \pm j\sqrt{\overbrace{\left(\frac{1}{LC}\right) - \left(\frac{R}{2L}\right)^2}^{\text{REAL}}}$$

$j = \sqrt{-1}$

- $v_C(t) = V_{DD} + a_1 e^{-(R/2L)t} e^{j\sqrt{(1/LC)-(R/2L)^2}\,t} +$

$$a_2 e^{-\underbrace{(R/2L)}_{\alpha}t} e^{-j\underbrace{\sqrt{(1/LC)-(R/2L)^2}}_{\omega_d}\,t}$$

$e^{j[\;]t}$

"ω"

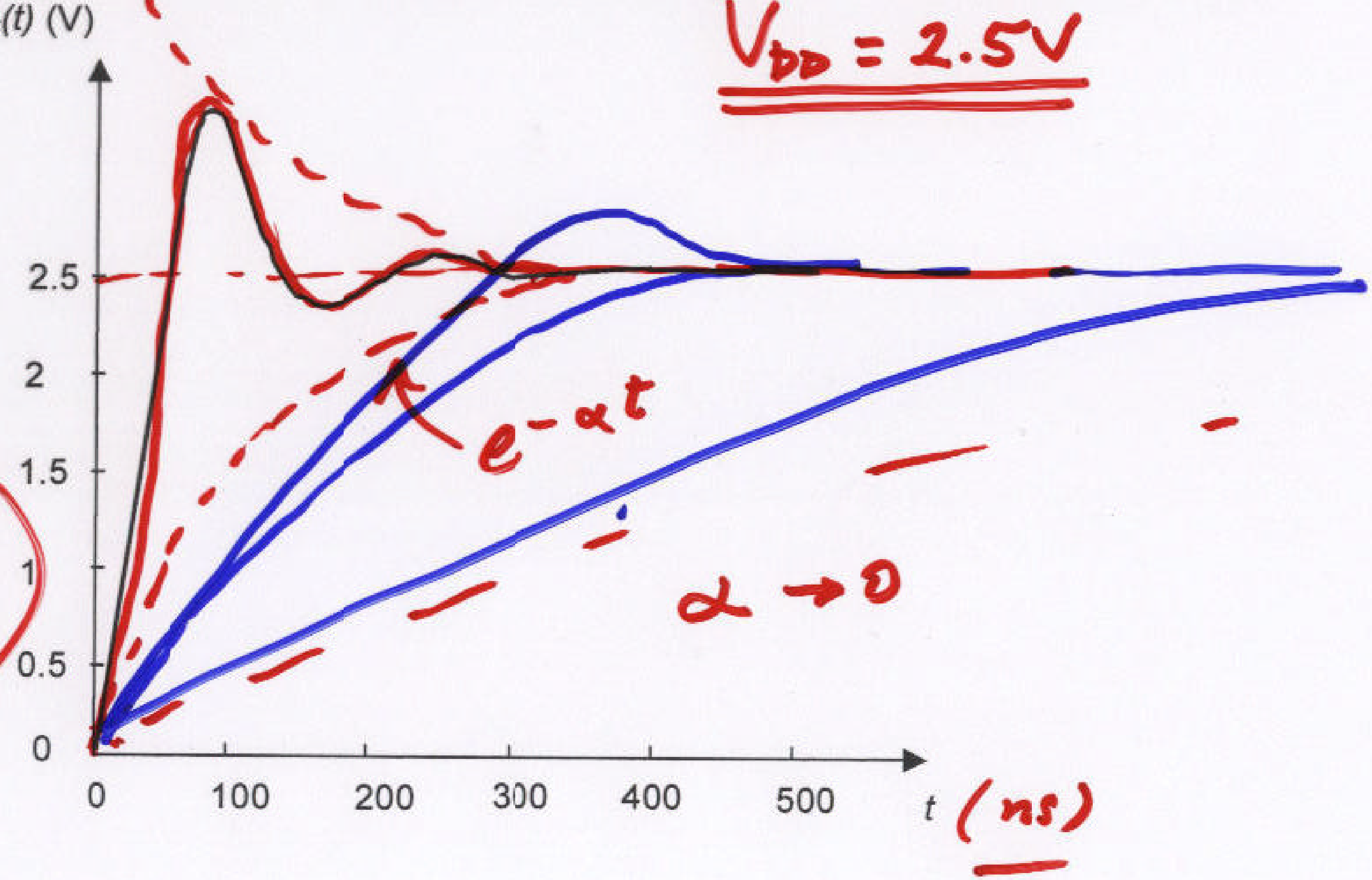
Form of solution …

$e^{j\omega t} + e^{-j\omega t}$

$v_C(t) = V_{DD} + [\;\cdot\;] e^{-\alpha t} \cos[\omega t + \phi]$

Qualitative Underdamped Waveform

$V_{DD} = 2.5V$



EE 141

UNDER-DAMPED.

Extreme Underdamped Case

$R \to 0.$

Exponential decay time is set by $\underline{\alpha = R/(2L)}$ $e^{-\alpha t}$

Small $\underline{R/L}$ → decay takes a long time and oscillation has a frequency that's nearly $\sqrt{1/(LC)}$

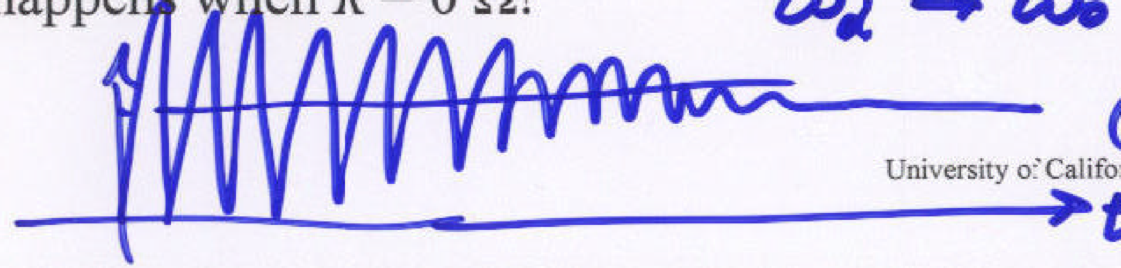
EE 142 ✓

Number of cycles during "ringdown" is

- $$N \approx \frac{(1/\alpha)}{\underbrace{(1/\sqrt{1/(LC)})}} = \frac{2L/R}{\sqrt{LC}} = \frac{2}{R}\sqrt{\frac{L}{C}}$$

What happens when $R = 0\ \Omega$?

$\omega_d \to \omega_0 = \frac{1}{\sqrt{LC}}$. $(R \to 0)$



ELECTROMECHANICAL OSCILLATOR.

R → 0.

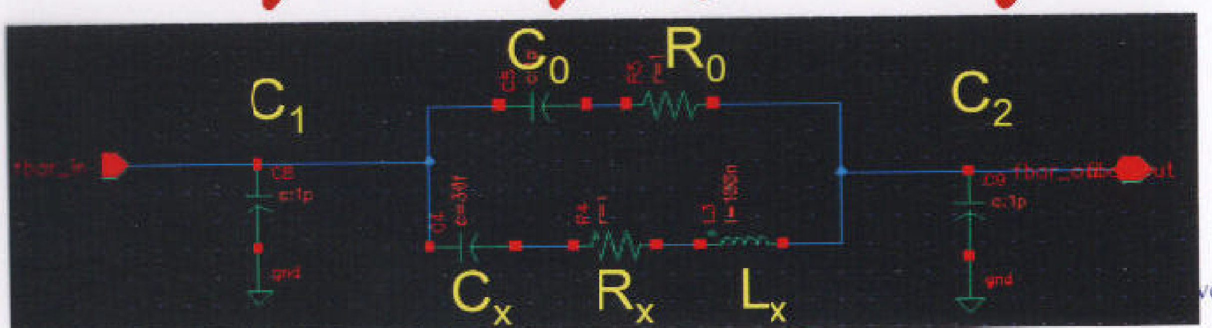
thin-Film Bulk Acoustic Resonator (FBAR)

RICH RUBY.

- Agilent Technologies
  IEEE ISSCC 2001,
  2 GHz resonator
- $N > 1000$

•Brian Otis, Jan Rabaey.
(BWRC): low-noise oscillator

•Equivalent Circuit:

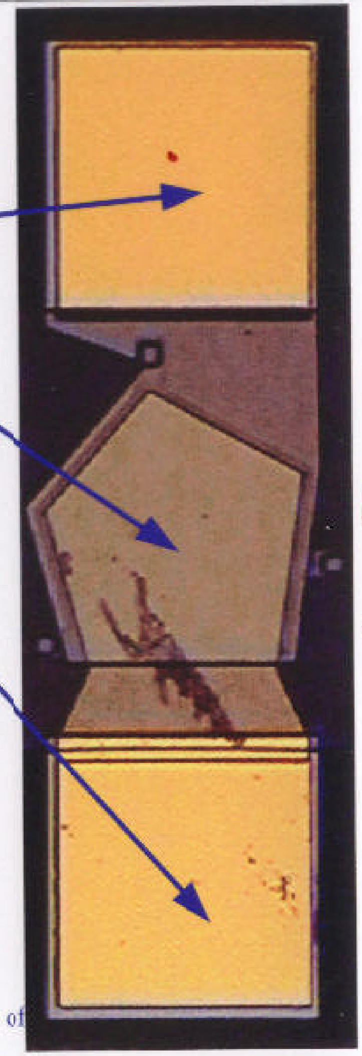


Drive Electrode

Thin Piezoelectric Film

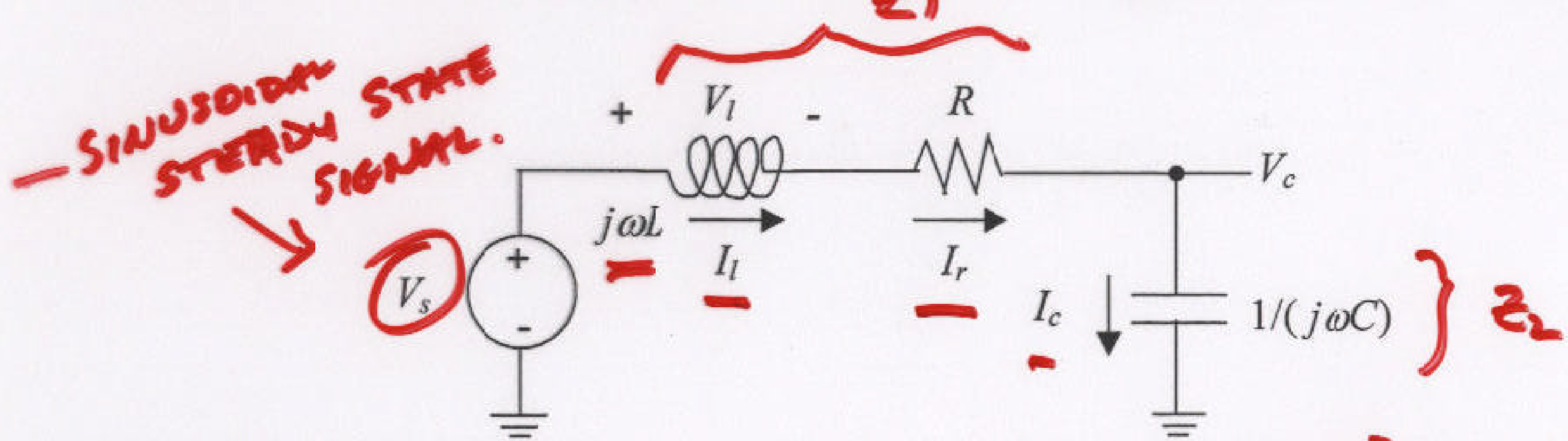
AlN

Sense Electrode



150 μm

Phasor Analysis of 2nd Order Circuit



→ Impedance divider:

$V_c = V_s \left(\frac{Z_2}{Z_1 + Z_2} \right)$

$$V_c = V_s \left[\frac{1/j\omega C}{(1/j\omega C) + R + (j\omega L)} \right] \left(\frac{j\omega C}{j\omega C} \right)$$

$j^2 \omega^2 LC$

Transfer Function

Simplifying:

$$H(j\omega)=\left[\frac{1}{1+j\omega RC-\omega^2 LC)}\right]$$

Define parameters: $\boxed{\omega_o = 1/\sqrt{LC}}$ $\tau = \underline{RC}$

$$H(j\omega)=\left[\frac{1}{1-(\omega/\omega_o)^2+j\omega\tau)}\right]$$

-NEW-
"STANDARD
FORM"

Limiting Cases: Magnitude and Phase

- Low frequency: $\omega << \omega_o$

$|H|_{dB} = 0\,dB \quad (1)$

$\angle H = 0°$

- High frequency: $\omega >> \omega_o$

$|H|_{dB} = \left| \frac{1}{\cancel{1} - (\omega/\omega_o)^2 + \cancel{j\omega\tau}} \right|_{dB}$

"$-40\,dB/dec$." "

$= \left| \frac{-1}{(\omega/\omega_o)^2} \right|_{dB}$.

Resonant frequency: $\omega = \omega_o$

$|H|_{dB} \Big|_{\omega = \omega_o} = \left| \frac{1}{\cancel{1} - \cancel{1} + j\omega_o\tau} \right|$

$= \left| \frac{1}{\omega_o\tau} \right|_{dB}$

$\angle H = -90°$.

$\angle H \Big|_{\omega \to \infty} = \angle \frac{1}{-(\omega/\omega_o)^2}$

$= \angle -(\cdot)$

TRIG!

$= \pm 180°$.

Inductor-Capacitor "Tuning"

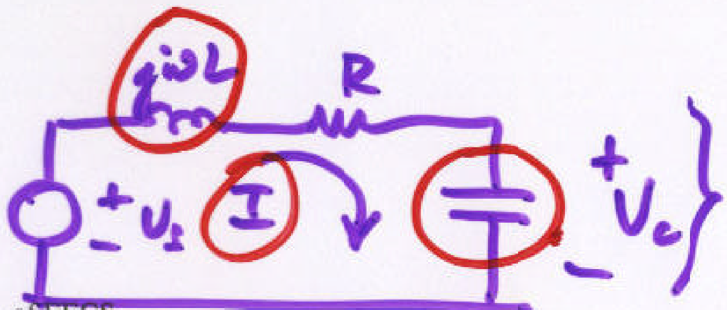
At resonance, the impedance of the capacitor cancels the impedance of the inductor → phasor current is maximum and capacitor voltage peaks

FALL 2001

- How "sharp" or "narrow" is the resonance?

FOR $\Delta\omega$ = WIDTH AT 1/2 PEAK OF TRANSFER FUNCTION

- Define the quality factor $Q = \dfrac{\omega_o}{\Delta\omega} \longrightarrow \boxed{Q = \dfrac{1}{\omega_o \tau}}$.



$\omega = \omega_0 = \dfrac{1}{\sqrt{LC}}$. N.

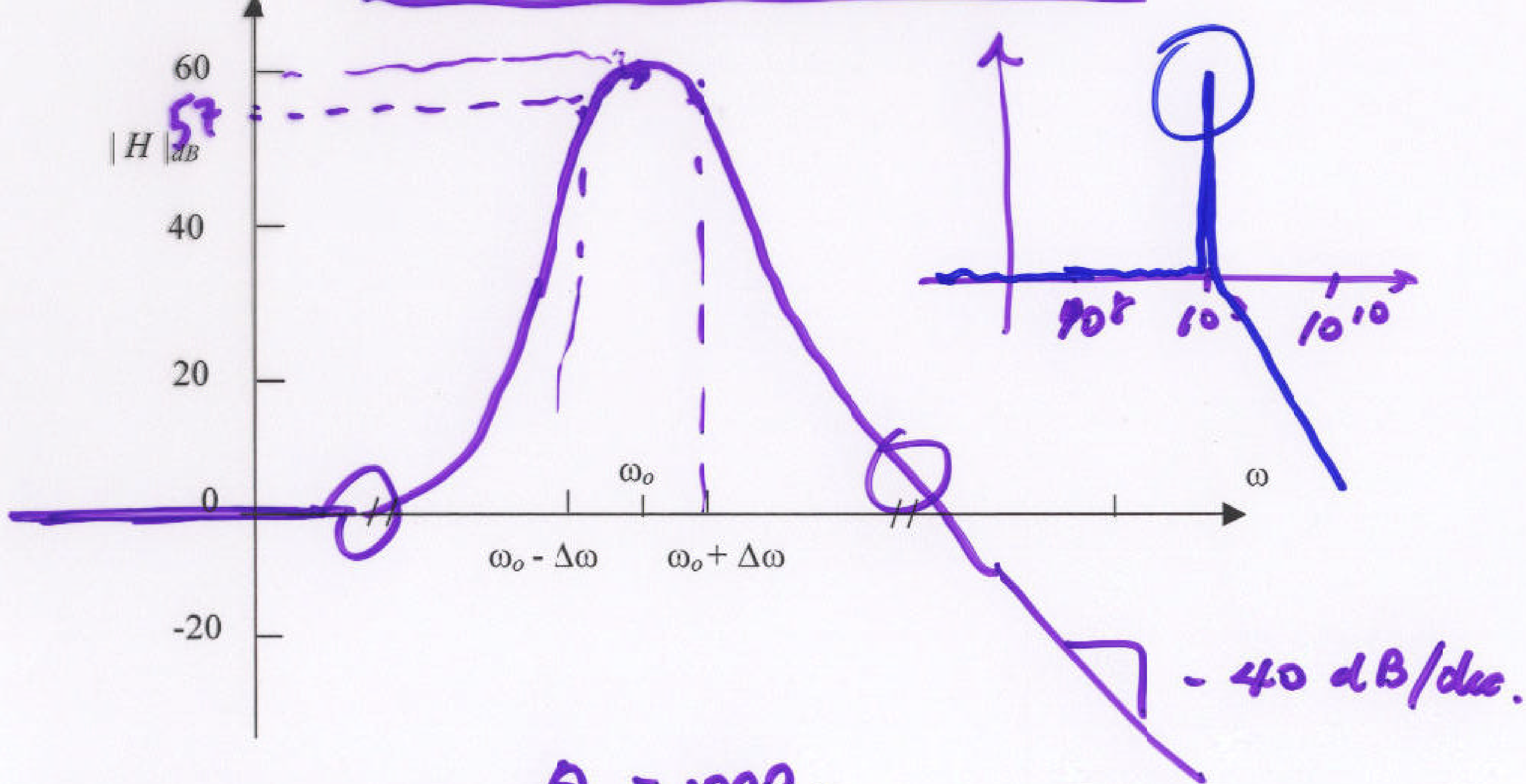
$j\omega_0 L = j\dfrac{1}{\sqrt{LC}} \cdot L = j\sqrt{\dfrac{L}{C}}$

$Z|_{\omega_0} = \cancel{j\omega_0 L} + R + \cancel{\dfrac{1}{j\omega_0 C}} = R.$

$\omega = \omega_0 \ldots \quad \frac{1}{j\omega_0 C} = \frac{1}{j\frac{1}{\sqrt{LC}}\cdot C} = -j\,\frac{1}{\sqrt{\frac{C}{L}}}$

$= \underline{\underline{-j\sqrt{\frac{L}{C}}}}$

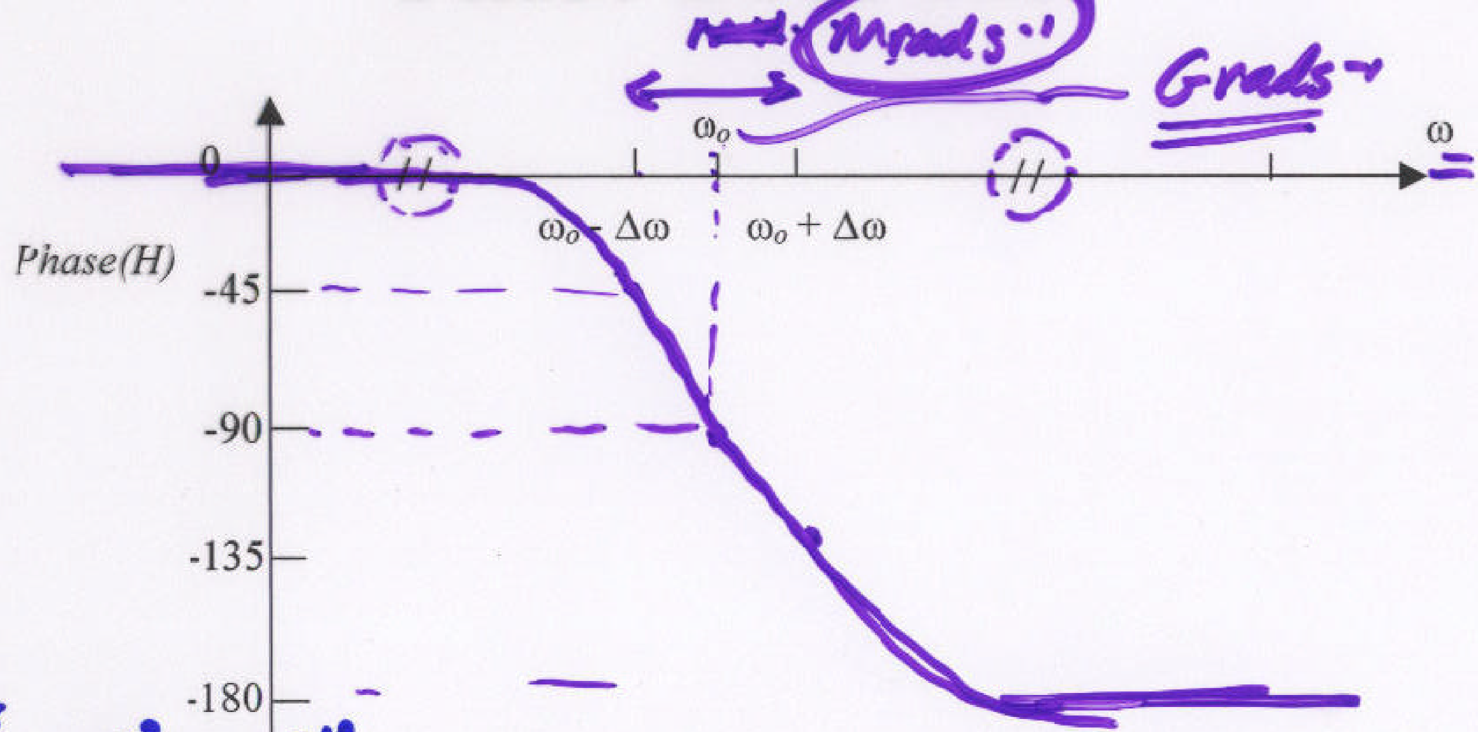
Magnitude Bode Plot

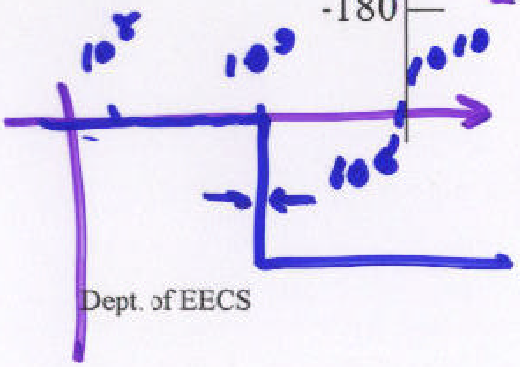


$Q = 1000.$

$|H| = |Q| = 60\,dB$

Phase Bode Plot





$$\Delta\omega = \frac{\omega_0}{Q} \quad \cdots$$

FBAR

$Q = 1000$

$\omega_0 = 2\pi(2\times10^9)\,\text{rads}^{-1}$