Lecture 9: Diffusion, Electrostatics review, and Capacitors

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Context

- In the last lecture, we looked at the carriers in a neutral semiconductor, and drift currents.
- In this lecture, we will continue to study transport—the motion of carriers due to diffusion, and the influence of charge distributions on the electric fields.
  - Review of Electrostatics
  - Diffusion
  - IC MIM Capacitors
- In the next lecture, we will study P-N diodes.

Electrostatics Review

- The force between any two charges is given by Coulomb’s law:
  \[ \vec{F} = \hat{r} \cdot \frac{4\pi q_1 q_2}{4\pi r^2} \]
- Where \( \hat{r} \) is unit vector in the direction away from the other charge.
- Since Maxwell’s equations are linear, we can add up all the forces from other charges, and define the electric field at a point:
  \[ \vec{E} = \frac{1}{\varepsilon} \sum_{i=1}^{n} \vec{F}_{1n} = \sum_{i=1}^{n} \hat{e}_{1n} \frac{q_2}{4\pi\varepsilon r^2} \]

Electrostatic fields

- Since we are going to be dealing with large numbers of charges, we can use a continuum model:
  - The charges are given as a smooth density:
    \[ \rho(\vec{r}, t) \text{ Coulombs/cm}^3 \]
  - The electric field is a smooth vector field which diverges from positive charge, and converges on negative charges:
    \[ \nabla \cdot \vec{E}(\vec{r}, t) = \frac{\rho(\vec{r}, t)}{\varepsilon} \]
  - Which is one of Maxwell’s equations.
**Gauss’s Law**

- Gauss’s Law states that the total amount of E field (flux) leaving a volume is equal to the net charge enclosed.

\[
\int S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\varepsilon}
\]

Recall:

\[
\int \int V \mathbf{E} \cdot dV = \frac{1}{\varepsilon} \int \int V \rho \, dV = \frac{Q}{\varepsilon}
\]

**Electrostatic Potential**

- The electric field (force) is related to the potential (energy):

\[
E = -\frac{d\phi}{dx}
\]

The E field is the (-) slope of the potential!

- Negative sign says that field lines go from high potential points to lower potential points (negative slope).

- Note: Electrons “float” to a high potential point:

\[
F_x = qE = -\frac{d\phi}{dx}
\]

**Electrostatics in 1D**

- In EE105, we are almost always going to use 1-D models, so it simplifies:

\[
\nabla \cdot \mathbf{E} = \frac{dE}{dx} = \frac{\rho}{\varepsilon}
\]

- Consider a uniform charge distribution

\[
E(x) = E(x_0) + \int_{x_0}^{x} \frac{\rho(x')}{\varepsilon} dx'
\]

**More Potential**

- Integrating this basic relation, we have that the potential is the integral of the field:

\[
\phi(x) - \phi(x_0) = -\int_{x_0}^{x} E(x') dx'
\]

- In 1D, this is a simple integral:

\[
\phi(x) - \phi(x_0) = -\int_{x_0}^{x} E(x') dx'
\]

- Since the derivative of the E field is the charge, we can integrate again to get Poisson’s equation in 1D:

\[
\frac{d^2 \phi(x)}{dx^2} = \frac{\rho(x)}{\varepsilon}
\]
Boundary Conditions

- Potential must be a continuous function. If not, the fields (forces) would be infinite.
- Electric fields need not be continuous. We have already seen that the electric fields diverge on charges. In fact, across an interface we have:

\[ \int e E \cdot dS = -\varepsilon_1 E_1 S + \varepsilon_2 E_2 S = Q_{\text{inside}} \]

\[ Q_{\text{inside}} \rightarrow 0 \]

\[ -\varepsilon_1 E_1 S + \varepsilon_2 E_2 S = 0 \]

\[ \frac{E_1}{\varepsilon_1} = \frac{E_2}{\varepsilon_2} \]

- Field discontinuity implies charge density at surface!

Note: Band edge diagrams

- We will often draw a diagram of the valence and the conduction band edges as a function of position.
- The energy at the band edge corresponds to the potential energy that an electron has (which is the negative of the electrostatic potential). Thus the slope of the band edge with distance is the electric field.

Diffusion

- Diffusion occurs when there exists a concentration gradient.
- In the figure below, imagine that we fill the left chamber with a perfume at temperature \( T \).
- If we suddenly remove the divider, what happens?
- The perfume will fill the entire volume of the new chamber. How does this occur?
Diffusion

- Even though there is no force acting on the perfume molecules, because there are more on the left than on the right, their random motions take more from the left to the right than are going from right to left.
- Electrons and holes do the same thing, but since they are charged, they carry current with them.
- Diffusion moves particles in addition to the motion from electric forces.

Diffusion (cont)

- The net motion of gas molecules to the right chamber was due to the concentration gradient.
- Diffusion will lead to a net flow of particles as long as the distribution of particles is not uniform.
- Diffusion causes a flow of particles from places of high concentration to places of lower concentration.

Diffusion Equations

- Assume that the mean free path is \( \lambda \).
- Find flux of carriers crossing \( x=0 \) plane.

\[
\text{Flux} = \frac{1}{2} v_0 \left( n(-\lambda) - n(\lambda) \right)
\]
\[
\text{Flux} = \frac{1}{2} v_0 \left( n(0) - \lambda \frac{dn}{dx} \right) - \left( n(0) + \lambda \frac{dn}{dx} \right)
\]
\[
\text{Flux} = -v_0 \Delta \frac{dn}{dx}
\]
\[
\text{Flux} = -D \frac{dn}{dx}
\]

Where the diffusion constant is defined:

\[
D_n = \frac{\text{Flux}}{\frac{dn}{dx}} = v_0 \lambda
\]
Relation between carrier Diffusion and carrier Mobility

- The motion of particles under the influence of a force (like an E field) and the motion due to concentration gradients are related to each other.
- For example, let's look at a semiconductor in an electric field, but also at thermal equilibrium:

Thermal Equilibrium

- A couple fundamental principles about thermal equilibrium:
  - The energy that electrons are filled up to (the Fermi level) is the same everywhere.
  - The current is zero at all points.
- So when we look at this:

Einstein Relation

- The average thermal velocity is given by $kT$
- The mean free time is
- $J = qv_n \frac{dn}{dx} = q \left( \frac{kT}{q} \mu_n \right) \frac{dn}{dx}$
- $D_n = \left( \frac{kT}{q} \right) \mu_n$

The Einstein relation (diffusion)

- Since the diffusion process has a fundamental relationship to the mobility in an electric field, we can find the diffusion constant in terms of the mobility $\mu$.

$$D_n = \left( \frac{kT}{q} \right) \mu_n$$

- We can do a rough derivation for the case of our simple particle model (next slide)
Total Current and Boundary Conditions

- The total current is given by the sum of drift and diffusion:
  \[ J = J_{\text{drift}} + J_{\text{diff}} = q\mu_n nE + qD \frac{dn}{dx} \]

- In resistors, the carriers are approximately uniform and the second term is nearly zero.

- In metals, there are a very large number of carriers, in very uniform concentration, and the conduction current is quite linear with \( E \) (ohmic).

Review of Capacitors

\[ \int E \cdot dl = E_0 t_{ox} = V_s \quad \Rightarrow \quad E_0 = \frac{V_s}{t_{ox}} \]

\[ \int E \cdot dS = E_0 A = \frac{Q}{\varepsilon} \quad \Rightarrow \quad \frac{V_s}{t_{ox}} A = \frac{Q}{\varepsilon} \quad \Rightarrow \quad C = \frac{A E}{t_{ox}} \]

IC MIM Capacitor

- By forming a thin oxide and metal (or polysilicon) plates, a capacitor is formed.
- Contacts are made to top and bottom plate.
- Parasitic capacitance exists between bottom plate and substrate (Parasitic means that it’s there whether we want it or not!)

Capacitor Q-V Relation

- The total charge on each plate is linearly related to voltage (+Q on top plate, -Q on bottom plate).
- Charge density is a delta function at surface (for metals).
**A Non-Linear Capacitor**

- We’ll soon meet capacitors that have a non-linear Q-V relationship.
- If plates are not metal, the charge density can penetrate into surface.

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**Introduction to small signal models**

- For a non-linear capacitor, we have
  \[ Q = f(V_c) \neq CV_c \]
- We can’t identify a capacitance.
- Imagine we apply a small signal on top of a bias voltage:
  \[ Q = f(V_c + \Delta V) \approx f(V_c) + \frac{df(V_c)}{dV_c} \Delta V_c \]
- The incremental charge is therefore:
  \[ Q = Q_c + q \approx f(V_c) + \frac{df(V_c)}{dV_c} \Delta V_c \]

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**Small Signal Capacitance**

- Break the equation for total charge into two terms:

\[ Q = Q_c + q \approx f(V_c) + \frac{df(V_c)}{dV_c} V_c \]

**Incremental Charge**

\[ q = \frac{df(V_c)}{dV_c} V_c \]

**Constant Charge**

\[ C = \frac{df(V_c)}{dV_c} V_c \]

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**Example of Non-Linear Capacitor**

- Next lecture we’ll see that for a PN junction, the charge is a function of the reverse bias:
  \[ Q_j(V) = -qN_x \sqrt{1 - \frac{V}{\phi_b}} \]

**Voltage Across NP Junction**

**Charge At N Side of Junction**

**Constants**

- Small signal capacitance:
  \[ C_j(V) = \frac{dQ_j}{dV} = \frac{qN_x}{2\phi_b} \sqrt{1 - \frac{V}{\phi_b}} = \frac{C_j}{\sqrt{1 - V/\phi_b}} \]
Carrier Concentration and Potential

- In thermal equilibrium, there are no external fields and we thus expect the electron and hole current densities to be zero:

\[ J_s = 0 = qn_s \mu_e E_v + qD_n \frac{dn}{dx} \]

\[ \frac{dn}{dx} = \left( \frac{\mu_e}{D_n} \right) n_s E_v = \left( \frac{q}{kT} \right) n_i \frac{d\phi}{dx} \]

\[ d\phi = \left( \frac{kT}{q} \right) \frac{dn}{n_0} = V_a \frac{dn}{n_0} \]

Carrier Concentration Versus Potential

- The carrier concentration is thus a function of potential

\[ n(x) = n_i e^{\phi(x)/V_a} \]

- Check that for zero potential, we have intrinsic carrier concentration (reference).

- If we do a similar calculation for holes, we arrive at a similar equation

\[ p(x) = n_i e^{\phi(x)/V_a} \]

- Note that the law of mass action is upheld

\[ n(x)p(x) = n_i^2 e^\phi(x)/2V_a \]

The Doping Changes Potential

- Due to the log nature of the potential, the potential changes linearly for exponential increase in doping:

\[ \phi(x) = V_a \ln \frac{n(x)}{n_i(x)} = 26 \text{mV} \ln \frac{n(x)}{n_i(x)} = 26 \text{mV} \ln 10 \frac{n_i(x)}{10^{10}} \]

\[ \phi(x) \approx 60 \text{mV} \log \frac{n_i(x)}{10^{10}} \]

\[ \phi(x) \approx -60 \text{mV} \log \frac{p_i(x)}{10^{10}} \]

- Quick calculation aid: For a p-type concentration of 10^{16} cm^{-3}, the potential is -360 mV

- N-type materials have a positive potential with respect to intrinsic Si
PN Junctions: Overview

- The most important device is a junction between a p-type region and an n-type region.
- When the junction is first formed, due to the concentration gradient, mobile charges transfer near junction.
- Electrons leave n-type region and holes leave p-type region.
- These mobile carriers become minority carriers in new region (can’t penetrate far due to recombination).
- Due to charge transfer, a voltage difference occurs between regions.
- This creates a field at the junction that causes drift currents to oppose the diffusion current.
- In thermal equilibrium, drift current and diffusion must balance.

PN Junction Fields

- Consider the PN junction in thermal equilibrium.
- Again, the currents have to be zero, so we have

\[
J_n = 0 = qn_0\mu_n E_n + qD_n \frac{dn_n}{dx}
\]

\[
q\mu_n E_n = qD_n \frac{dn_n}{dx}
\]

\[
E_n = \frac{-D_n \frac{dn_n}{dx}}{n_0 \mu_n} = -\frac{kT}{q} \frac{1}{n_0} \frac{dn_n}{dx}
\]

\[
E_n = \frac{D_n \frac{dp_n}{dx}}{n_0 \mu_n} = -\frac{kT}{q} \frac{1}{n_0} \frac{dp_n}{dx}
\]

Total Charge in Transition Region

- To solve for the electric fields, we need to write down the charge density in the transition region:

\[
\rho_n(x) = q(p_n - n_b + N_p - N_n)
\]

- In the p-side of the junction, there are very few electrons and only acceptors:

\[
\rho_n(x) = q(p_n - N_n) \quad -x_p < x < 0
\]

- Since the hole concentration is decreasing on the p-side, the net charge is negative:

\[
N_n > p_n \quad \rho_n(x) < 0
\]
Charge on N-Side

- Analogous to the p-side, the charge on the n-side is given by:
  \[ \rho_n(x) \approx q(-n_i + N_n) \quad 0 < x < x_{soi} \]

- The net charge here is positive since:
  \[ N_n > n_i \quad \rho_n(x) > 0 \]