* Website, courses, up
* Sign up Piazza
* Lab Discussion start next week

EE105
Microelectronic Devices and Circuits

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Linear Time-Invariant (LTI) System

• Response of a system

\[x(t) \rightarrow H \rightarrow y(t)\]

• The system is linear if

\[a_1 x_1(t) + a_2 x_2(t) \rightarrow a_1 y_1(t) + a_2 y_2(t)\]

• The system is time-invariant if

\[x(t + T) \rightarrow H \rightarrow y(t + T)\]

\[H \rightarrow a_1 y_1(t) + a_2 y_2(t)\]
What’s Nice about LTI System?

• Can use superposition
• Easy conversion between time and frequency response
• Most systems in real life are LTI systems
  – Focus of this class
Example: Low Pass Filter (LPF)

\[
\omega = 2\pi f \quad \left[ \text{rad/s} \right] \quad \left[ \text{Hz} \right] = \left[ \frac{1}{s} \right]
\]

- **Input signal:**
  \[v_s(t) = V_s \cos(\omega t)\]

- **We know that:**
  \[v_o(t) = K \cdot V_s \cos(\omega t + \phi)\]

\[v_o(t) = v_s(t) - i(t)R\]
\[i(t) = C \frac{dv_o}{dt}\]
\[v_o(t) = v_s(t) - RC \frac{dv_o}{dt}\]
\[v_s(t) = v_0(t) + \tau \frac{dv_o}{dt}\]

Phase shift

Amp scale

\[\text{Amp scale}\]

\[\text{Phase shift}\]

\[\omega = 2\pi f \quad \left[ \text{rad/s} \right] \quad \left[ \text{Hz} \right] = \left[ \frac{1}{s} \right]\]
Exponential Representation

• Euler’s Theorem

\[ e^{j\omega t} = \cos(\omega t) + j \sin(\omega t) \]

• \( \sin(\omega t) \) and \( \cos(\omega t) \) can be represented by linear combination of complex exponential:

\[ \cos(\omega t) = \frac{1}{2} (e^{j\omega t} + e^{-j\omega t}) \]

\[ \sin(\omega t) = \frac{1}{2j} (e^{j\omega t} - e^{-j\omega t}) \]
Magic: Turn Diff Eq into Algebraic Eq

• Integration and differentiation are trivial with complex numbers:

\[
\frac{d}{dt} e^{j\omega t} = j\omega e^{j\omega t} \quad \int e^{j\omega \tau} d\tau = \frac{1}{i\omega} e^{j\omega t}
\]

• Any ODE is now trivial algebraic manipulations … in fact, we’ll show that you don’t even need to directly derive the ODE by using phasors

• The key is to observe that the current/voltage relation for any element can be derived for complex exponential excitation
Solving LPF with Phasors

• Let’s excite the system with a complex exp:

\[ v_s(t) = v_0(t) + \tau \frac{dv_0}{dt} \]

\[ v_s(t) = V_s e^{j\omega t} \]

\[ v_o(t) = |V_0| e^{j(\omega t + \phi)} = V_0 e^{j\omega t} \]

\[ V_s e^{j\omega t} = V_0 e^{j\omega t} + \tau \cdot j\omega \cdot V_0 e^{j\omega t} \]

\[ V_s = V_0 (1 + j\omega \cdot \tau) \]

\[ \frac{V_0}{V_s} = \frac{1}{(1 + j\omega \cdot \tau)} \]

use \( j \) to avoid confusion

\[ V_0 = |V_o| e^{j\phi} \]

Easy!!!
Magnitude and Phase Response

• The system is characterized by the complex function

\[ H(\omega) = \frac{V_0}{V_s} = \frac{1}{1 + j \omega \tau} \]

• The magnitude and phase response match our previous calculation:

\[ |H(\omega)| = \left| \frac{V_0}{V_s} \right| = \frac{1}{\sqrt{1 + (\omega \tau)^2}} \]

\[ \angle H(\omega) = -\tan^{-1} \omega \tau \]
Why did it work?

- Again, the system is **linear**: 
  \[ y = L(x_1 + x_2) = L(x_1) + L(x_2) \]

- To find the response to a sinusoid, we can find the response to \( e^{i\omega t} \) and \( e^{-i\omega t} \) and sum the results:

\[
\begin{align*}
  e^{i\omega t} & \rightarrow \text{LTI System } H \\
  & \rightarrow |H(\omega)| e^{i(\omega t + \phi_1)} \\
  \downarrow & \\
  e^{-i\omega t} & \rightarrow \text{LTI System } H \\
  & \rightarrow |H(-\omega)| e^{i(-\omega t + \phi_2)} \\
  e^{i\omega t} + e^{-i\omega t} & = \frac{2}{\cos(\omega t)} \\
  & \rightarrow \text{LTI System } H \\
  & \rightarrow H(\omega)e^{i\omega t} + H(-\omega)e^{-i\omega t}
\end{align*}
\]
• Since the input is real, the output has to be real:

\[ y(t) = \frac{H(\omega)e^{i\omega t} + H(-\omega)e^{-i\omega t}}{2} \]

• That means the second term is the conjugate of the first:

\[ H(-\omega) = H(\omega)^* \]

\[ |H(-\omega)| = |H(\omega)| \] (even function)

\[ \angle H(-\omega) = -\angle H(\omega) = -\phi \] (odd function)

• Therefore the output is:

\[ y(t) = \left| H(\omega) \right| \left( e^{i(\omega t + \phi)} + e^{-i(\omega t + \phi)} \right) \]

\[ x(t) = \cos(\omega t) = \left| H(\omega) \right| \cos(\omega t + \phi) \]
Phasors

- With our new confidence in complex numbers, we go full steam ahead and work directly with them … we can even drop the time factor $e^{i\omega t}$ since it will cancel out of the equations.
- Excite system with a phasor: $\tilde{V}_1 = V_1 e^{j\phi_1}$
- Response will also be phasor: $\tilde{V}_2 = V_2 e^{j\phi_2}$
- For those with a Linear System background, we’re going to work in the frequency domain – This is the Laplace domain with $s = j\omega$
Capacitor I-V Phasor Relation

- Find the Phasor relation for current and voltage in a cap:

\[ i_c(t) = C \frac{dv_c(t)}{dt} \]
\[ v_c(t) = V_c e^{j\omega t} \]
\[ i_c(t) = I_c e^{j\omega t} \]

\[ I_c e^{j\omega t} = C \frac{d}{dt} [V_c e^{j\omega t}] \]
\[ C V_c \frac{d}{dt} e^{j\omega t} = j\omega CV_c e^{j\omega t} \]
\[ I_c e^{j\omega t} = j\omega CV_c e^{j\omega t} \]
\[ I_c = j\omega CV_c \]

90° Phase Shift between current & voltage

Impedence

\[ \frac{V_c}{I_c} = \frac{1}{j\omega C} = Z \]

Resistor
\[ V = IR \]
\[ \frac{V}{I} = R \]

\[ [E] = [\frac{V}{I}] = [S] \]
Inductor I-V Phasor Relation

- Find the Phasor relation for current and voltage in an inductor:

\[
v(t) = L \frac{di(t)}{dt}
\]

\[
i(t) = I e^{j\omega t}
\]

\[
v(t) = V e^{j\omega t}
\]

\[
Ve^{j\omega t} = L \frac{d}{dt} [I e^{j\omega t}]
\]

\[
LI \frac{d}{dt} e^{j\omega t} = j\omega LI e^{j\omega t}
\]

\[
Ve^{j\omega t} = j\omega LI e^{j\omega t}
\]

\[
V = j\omega LI
\]

\[
\frac{V}{I} = j\omega L = \mathcal{Z}_L
\]

\[
\begin{bmatrix} 1 \\ 5 \end{bmatrix} [H] = [\Omega]
\]
Impede the Currents!

• Suppose that the “input” is defined as the voltage of a terminal pair (port) and the “output” is defined as the current into the port:

\[ v(t) = V e^{j\omega t} = |V| e^{j(\omega t + \phi_v)} \]
\[ i(t) = I e^{j\omega t} = |I| e^{j(\omega t + \phi_i)} \]

• The impedance \( Z \) is defined as the ratio of the phasor voltage to phasor current (“self” transfer function)

\[ Z(\omega) = H(\omega) = \frac{V}{I} = \left| \frac{V}{I} e^{j(\phi_v - \phi_i)} \right| \]
Admit the Currents!

• Suppose that the “input” is defined as the current of a terminal pair (port) and the “output” is defined as the voltage into the port:

\[ v(t) = V e^{j\omega t} = |V| e^{j(\omega t + \phi_v)} \]
\[ i(t) = I e^{j\omega t} = |I| e^{j(\omega t + \phi_i)} \]

• The admittance \( Y \) is defined as the ratio of the phasor current to phasor voltage (“self” transfer function)

\[ Y(\omega) = H(\omega) = \frac{I}{V} = \left| \frac{I}{V} \right| e^{j(\phi_i - \phi_v)} \]

**Admittance** \( [\text{Y}] = [\text{S}] \), mho

\( [\text{S}] \), Ohm
Voltage and Current Gain

- The voltage (current) gain is just the voltage (current) transfer function from one port to another port:

\[ G_v(\omega) = \frac{V_2}{V_1} = \frac{V_2}{V_1} e^{j(\phi_2 - \phi_1)} \]

- If \(|G| > 1\), the circuit has voltage (current) gain
- If \(|G| < 1\), the circuit has loss or attenuation
Transimpedance/admittance

- Current/voltage gain are unit-less quantities
- Sometimes we are interested in the transfer of voltage to current or vice versa

\[ V: \text{Gain} = \frac{V_2}{V_1} \]
\[ I: \text{Gain} = \frac{I_2}{I_1} \]

\[ J(\omega) = \frac{V_2}{I_1} = \left| \frac{V_2}{I_1} \right| e^{j(\phi_2 - \phi_1)} \]

\[ K(\omega) = \frac{I_2}{V_1} = \left| \frac{I_2}{V_1} \right| e^{j(\phi_2 - \phi_1)} \]
Direct Calculation of $H$ (no DEs)

- To directly calculate the transfer function (impedance, trans-impedance, etc) we can generalize the circuit analysis concept from the “real” domain to the “phasor” domain.

- With the concept of impedance (admittance), we can now directly analyze a circuit without explicitly writing down any differential equations.

- Use KVL, KCL, mesh analysis, loop analysis, or node analysis where inductors and capacitors are treated as complex resistors.
LPF Example: Again!

• Instead of setting up the DE in the time-domain, let’s do it directly in the frequency domain.

• Treat the capacitor as an impedance:

\[ Z_R = R \]

\[ Z_C = \frac{1}{j\omega C} \]

\[ V_o = V_c \cdot \frac{Z_c}{Z_R + Z_c} \]

\[ H = \frac{V_o}{V_c} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \]
Bode Plots

• Simply the log-log plot of the magnitude and phase response of a circuit (impedance, transimpedance, gain, ...)

• Gives insight into the behavior of a circuit as a function of frequency

• The “log” expands the scale so that breakpoints in the transfer function are clearly delineated
Frequency Response of Low-Pass Filters

\[ H(\omega) = \frac{1}{j\omega C} \]

\[ T(\omega) = \frac{1}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\omega / \omega_0} \]

\[ \omega_0 = \frac{1}{RC} = \frac{1}{\tau} \]

\[ |T(\omega)| = \frac{1}{\sqrt{1 + (\omega / \omega_0)^2}} \]

\[ \angle T(\omega) = -\tan^{-1} \left( \frac{\omega}{\omega_0} \right) \]

\[ \omega_{3dB} = \omega_0 \quad [\text{rad/sec}] \]

\[ f_{3dB} = \frac{\omega_0}{2\pi} \quad [\text{Hz}] \]
Frequency Response of High-Pass Filters

\[
T(\omega) = \frac{R}{R + \frac{1}{j\omega C}} = \frac{1}{1 + \frac{1}{j\omega RC}} = 1 - j\omega_0 / \omega
\]

\[
\omega_0 = \frac{1}{RC}
\]

\[
|T(\omega)| = \frac{1}{\sqrt{1 + (\omega_0 / \omega)^2}}
\]

\[
\angle T(\omega) = \tan^{-1} \left( \frac{\omega_0}{\omega} \right)
\]

\[
\omega_{3dB} = \omega_0 \quad \text{[rad/sec]}
\]

\[
f_{3dB} = \frac{\omega_0}{2\pi} \quad \text{[Hz]}
\]
Example: High-Pass Filter

- Using the voltage divider rule:

\[ H(\omega) = \frac{j\omega L}{R + j\omega L} = \frac{j\omega L}{R} = \frac{1}{1 + j\omega \frac{L}{R}} \]

\[ H(\omega) = \frac{j\omega \tau}{1 + j\omega \tau} \]

\[ \omega \to \infty \quad |H| \to \left| \frac{j\omega \tau}{j\omega \tau} \right| = 1 \]

\[ \omega \to 0 \quad |H| \to \frac{0}{1 + 0} = 0 \]

\[ \omega = \frac{1}{\tau} \quad |H| = \left| \frac{j}{1 + j} \right| = \frac{1}{\sqrt{2}} \]
Approximate versus Actual Plot

- Approximate curve accurate away from breakpoint
- At breakpoint there is a 3 dB error
HPF Phase Plot

• Phase can be naturally decomposed as well:

\[ \angle H(\omega) = \angle \frac{j\omega \tau}{1 + j\omega \tau} = \angle j\omega \tau + \angle \frac{1}{1 + j\omega \tau} = \frac{\pi}{2} - \tan^{-1}(\omega \tau) \]

• First term is simply a constant phase of 90 degrees
• The second term is the arctan function
• Estimate arctan function:

\( \omega << \frac{1}{\tau} \)
\( \omega = \frac{1}{\tau} \)
\( \omega >> \frac{1}{\tau} \)
Power Flow

• The instantaneous power flow into any element is the product of the voltage and current: \( P(t) = i(t)v(t) \)

• For a periodic excitation, the average power is:

\[
P_{av} = \int_{T} i(\tau)v(\tau) d\tau
\]

• In terms of sinusoids we have

\[
P_{av} = \int_{T} |I| \cos(\omega t + \phi_i) |V| \cos(\omega t + \phi_v) d\tau
\]

\[
= |I| \cdot |V| \int_{T} (\cos \omega t \cos \phi_i - \sin \omega t \sin \phi_i) \cdot (\cos \omega t \cos \phi_v - \sin \omega t \sin \phi_v) d\tau
\]

\[
= |I| \cdot |V| \int_{T} d\tau \cos^2 \omega t \cos \phi_i \cos \phi_v + \sin^2 \omega t \sin \phi_i \sin \phi_v + c \sin \omega t \cos \omega t
\]

\[
= \frac{|I| \cdot |V|}{2} (\cos \phi_i \cos \phi_v + \sin \phi_i \sin \phi_v) = \frac{|I| \cdot |V|}{2} \cos(\phi_i - \phi_v)
\]
Power Flow with Phasors

\[ P_{av} = \frac{|I| \cdot |V|}{2} \cos(\phi_i - \phi_v) \]

Power Factor

- Note that if \((\phi_i - \phi_v) = \frac{\pi}{2}\), then \(P_{av} = \frac{|I| \cdot V}{2} \cos \left( \frac{\pi}{2} \right) = 0\)

- Important: Power is a non-linear function so we can’t simply take the real part of the product of the phasors:
  \[ P \neq \text{Re}[I \cdot V] \]

- From our previous calculation:
  \[ P = \frac{|I| \cdot |V|}{2} \cos(\phi_i - \phi_v) = \frac{1}{2} \text{Re}[I \cdot V^*] = \frac{1}{2} \text{Re}[I^* \cdot V] \]
Summary

• Complex exponentials are eigen-functions of LTI systems
  – Steady-state response of LCR circuits are LTI systems
  – Phasor analysis allows us to treat all LCR circuits as simple “resistive” circuits by using the concept of impedance (admittance)

• Frequency response allows us to completely characterize a system
  – Any input can be decomposed into either a continuum or discrete sum of frequency components
  – The transfer function is usually plotted in the log-log domain (Bode plot) – magnitude and phase
  – Location of poles/zeros is key