Basic Coupled Mode Equations - Electric Field Gain

Generalization of The Amplifier Gain Equation

Basic Equation
\[ \frac{dP}{dz} = gP \]

Gain coefficient \( g \) \((m^{-1})\)

Solution
\[ P_{out} = P(L) = P_{in} e^{gL} = P_{in} G \]

Express this in terms of field amplitude.
To be able to treat phase sensitive devices

Basic Equation
\[ \frac{dE}{dz} = \frac{g}{\lambda} E \]

Let \( E = (E_0 e^{j\omega t} + \text{c.c.}) \) and separate into \( e^{j\omega t} \) and \( e^{-j\omega t} \) terms (\( E_0 \) = \( |E_0| \) \( e^{j\phi_0} \) complex)

Time average power \( P_{ave} = \frac{1}{2} \int_{-\infty}^{\infty} E^2 dt \)

Basic Equation
\[ \frac{dE_o}{dz} = \left( \frac{g}{\lambda} \right) E_o \]

Complex

\[ E_{out} = \left( E_{in} \right)^* e^{jG_L} \]

Time averaged output power \( \alpha \times E_{out} E_{out}^* \)

Thus \( P_{out,ave} = P_{in,ave} e^{jG_L} \)

Where \( G = \text{Re} \left( \frac{\alpha}{\lambda} \right) \)

Advantage: with \( \alpha \), one obtains the phase change as well as amplitude change (amplification).
Coupled Modes - A Model Switch or Directional Coupler

Two fibers (or optical waveguides which come into close proximity (separation d) over length L, the field of one guide (R-say) can penetrate the other guide (guide 2) and influence its field (S-say). Vice versa as well. Let the "coupling" be specified by a parameter $\kappa$. In analogy with a single gain equation, we now have two coupled equations:

1. $\frac{dR}{dz} = \kappa S + \frac{\hat{\kappa}}{2} R$
2. $\frac{dS}{dz} = \left( \kappa R + \frac{\hat{\kappa}}{2} S \right)$

Coupling terms.

Let there be no gain (for the moment). Thus $\hat{\kappa} = 0$.

Assume a lossless coupler. Then the total time-average power must not change $\frac{dP}{dz} = 0$.

But $P_{avg} = \frac{1}{2} \frac{d}{dz} \left( RR^* + SS^* \right)$. Thus multiply Eq(1) by $R^*$ and Eq(2) by $S^*$ and add the two resulting Eqns to obtain:

$\frac{d}{dz} (RR^* + SS^*) = \kappa (SR^* + RS^*) + \kappa^* (S^*R + R^*S)$

Thus $\kappa = -\kappa^*$ or $\kappa$ is imaginary (lossless).

Letting $\kappa = i\kappa$, the basic equations are:

1. $\frac{dR}{dz} = i\kappa S$
2. $\frac{dS}{dz} = i\kappa R$
Solutions For The Coupler.

1. \( \frac{dR}{dz} = iKR \quad \frac{ds}{dz} = iKS \)

From 2. \( \frac{d^2R}{dz^2} = iK \frac{ds}{dz} = (iK)^2 R = -K^2 R \)

Thus \( R = A \cos(kz) + B \sin(kz) \)
\( S = C \cos(kz) + D \sin(kz) \)

But From Eq(1) \(-Ak = iKb \) and \( kB = iKC \)

Thus \( R = A \cos(kz) + iC \sin(kz) \)
\( S = C \cos(kz) + iA \sin(kz) \)

Example: 3 dB Coupler

Let \((kL) = \left( \frac{\pi}{4} \right) \); \( C = 0 \) (No S-wave at input)

\[ R(L) = A \frac{1}{\sqrt{2}} \]
\[ S(L) = iA \frac{1}{\sqrt{2}} \]
\( \frac{\pi}{4} \) phase shift

2) Wavelength Multiplexer - Demultiplexer

Design coupling so that at \( \lambda_1 \) (wavelength) cross-over occurs, while at \( \lambda_2 \) it doesn't.

\( \lambda_1 \)
\( \lambda_2 \)
\( \frac{\pi}{2} \)
\( K_{\lambda_1} L = \frac{\pi}{2} \)
\( K_{\lambda_2} L = \pi \)

Value of \( K \)

For \( k^1 \ll k^2 \) \( kx \ll \frac{1}{L} \)

\[ K \approx \frac{1}{2} \frac{(\Delta n)(\frac{2\pi}{\lambda})}{g} \left( \frac{e^{-k^1(x-t)}}{10^1 \cdot 10^{-5}} \right) \]

Coupled Waveguides