Basic Equations for Free-Space Comm. Links

The Friis formula for a communication link is given by

\[ \frac{P_r}{P_t} = \frac{A_{er} A_{et}}{r^2 \lambda^2} \]

where

- \( P_r \) is the received power
- \( P_t \) is the transmitted power
- \( A_{er} \) is the effective area of the receiving antenna
- \( A_{et} \) is the effective area of the transmitting antenna
- \( r \) is the distance between the antennas
- \( \lambda \) is the wavelength.

a) Establish the validity of this expression for \( P_r / P_t \) based upon a simple diffraction argument (i.e., uncertainty relation).

1. For Gaussian \( D / 2 \) is the \( \frac{1}{e^2} \) intensity radius at the antenna \((z = 0)\).

\[ \text{Diffraction Angle} = \frac{\lambda}{D} \times \frac{2}{\pi} = \theta \quad \theta \text{ is the far field cone angle to } 0.6 \text{ times the peak intensity at } \theta = 0. \]

2. \( \Omega_A = 2\pi(1 - \cos\theta) = \pi\theta^2 = \pi\left(\frac{\lambda}{D}\right)^2 \left(\frac{2}{\pi}\right)^2 = \text{solid angle of antenna diffraction cone.} \]

3. \( P_t = \text{Power radiated into cone (Watts); Intensity at } r, S = \left(\frac{P_t}{\pi \theta^2 r^2}\right) \text{ (Watts/m}^2\) \]

4. \( P_r = \text{Received Power} = S \cdot A_{er} = \frac{P_t}{\pi \theta^2 r^2} A_{er} = \frac{P_t A_{er}}{\pi \left(\frac{\lambda}{D}\right)^2 \left(\frac{2}{\pi}\right)^2} \]

\[ = \frac{P_t \pi D^2}{\lambda^2 4r^2} A_{er} = \frac{P_t A_{et} A_{er}}{\lambda^2 r^2} \]

Thus the signal to thermal noise ratio = \( (P_t A_{et} A_{er}) / (\lambda^2 r^2 kT\Delta f) \)

b) Argue similarly that if the directivity of an antenna pattern is defined as \( \Omega_A = \frac{P(\theta, \varphi)_{\text{max}}}{P(\theta, \varphi)_{\text{avg}}} \) then \( D = \frac{4\pi}{\Omega_A} \) where \( \Omega_A \) is the solid angle of the main beam.
\[ P(\theta, \varphi)_{\text{max}} \Omega \Delta r^2 = \text{total power} = \frac{P(\theta, \varphi)_{\text{avg}}}{\Omega} \times 4\pi r^2 \]

\[ D \overset{\Delta}{=} \frac{P(\theta, \varphi)_{\text{max}}}{\frac{\Omega}{\Delta}} = \frac{4}{\Omega} \overset{\Delta}{=} \frac{\text{Gain}}{k} \quad k = 1 \text{ if lossless} \]

c) Using a diffraction argument show that in general

\[ D = \frac{4\pi}{\Omega} \frac{\Delta}{\theta^2} = \frac{4\pi}{\pi} \frac{4\pi^2(D/2)^2}{\lambda^2} = \frac{4\pi^2(D/2)^2}{\lambda^2} = 4\pi \frac{A_e}{\lambda^2} \]

\[ A_e = \pi \left( \frac{D}{2} \right)^2 \]

(Note: For radar, treat \( A_{er} \) in 4 as a cross-section for isotropic scattering, defined as \( \sigma \), and apply 4 a second time for the return path. Thus let \( P_{tr} \) be the radar return power. This is given by

\[ P_{tr} = \left( \frac{P_t A_{et} \sigma}{\lambda^2 r^2} \right) \]

For isotropic scattering the effective transmitting area is

\[ A_{et} = \frac{\lambda^2}{4\pi} (D = 1) \text{ ; then applying 4 above} \]

the radar signal received

\[ = \frac{P_{tr} A_{et} A_{et}}{\lambda^2 r^2} \sigma = P_t (A_{et})^2 \sigma / 4\pi \lambda^2 r^4 \]

Assumes transmitter antenna = receiver antenna