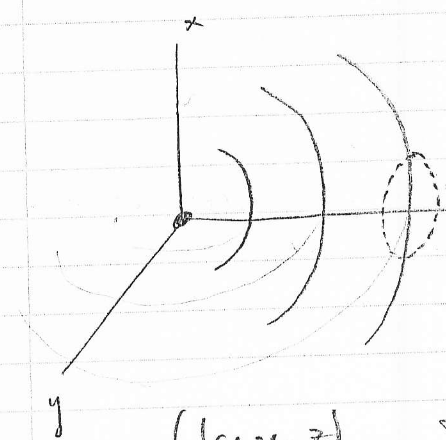
$$\frac{1}{q(z)} = \frac{1}{R(z)}$$

MORE ON "GAUSSIAN BEAMS"
- What are we really talking about here?



START: we wish to describe (mathematically) the spatial phase evolution of a spherical wave in a special location → close to the optic axis (small x^2+y^2) and far from the origin of the source (large z). This leads to the validity of the paraxial approx.

$$E_{spherical}(x,y,z,t) = \frac{A_0}{r} e^{\overbrace{ikr}^{S(x,y,z)}-i\omega t} \quad \text{where } r \equiv \sqrt{x^2+y^2+z^2}$$

look at r; assume $z \gg \sqrt{x^2+y^2}$

$$r = \sqrt{x^2+y^2+z^2} = z\sqrt{1+\frac{x^2+y^2}{z^2}} = z\underbrace{\left(1+\frac{x^2+y^2}{2z^2}+\frac{1}{8}\left[\frac{x^2+y^2}{z^2}\right]^2+\ldots\right)}_{\text{TAYLOR EXPANSION.}}$$

For $x^2+y^2 \ll z^2$, $\frac{x^2+y^2}{z^2} \ll 1 \quad \therefore \quad \frac{1}{8}\left[\frac{x^2+y^2}{z^2}\right]^2 \ll \frac{1}{2}\left[\frac{x^2+y^2}{z^2}\right]$

so neglect this term & all other H.O.T. IN THE PHASE and just let $r \cong z$ in the amplitude.

$$E_{\substack{spherical \\ x^2+y^2 \gg z^2}}(x,y,z,t) \cong \frac{A_0}{z} e^{ikz+\frac{ik}{2}\frac{x^2+y^2}{z}-i\omega t}$$

$$\cong \frac{A_0}{z} e^{ikz-i\omega t} \exp\left\{\frac{ik}{2z}(x^2+y^2)\right\}$$

QUADRATIC PHASE APPROX TO A SPHERICAL PHASEFRONT

$$\therefore \quad R_r(z) = \frac{z^2+z_0^2}{z}$$

$$\frac{\lambda}{\pi n W^2(z)} = \frac{z_0}{z^2+z_0^2}$$

$$W^2(z) = \frac{\lambda}{\pi n}\frac{(z^2+z_0^2)}{z_0}$$

$$W_0^2 = W^2(z=0)$$

$$W^2(z) =$$

Mathematically

Why? adding an imaginary part to z allows us to ~~[illegible]~~ create a field whose amplitude can change ~~[illegible]~~ in z and in x & y (which we know is true in reality)

So, make this "weird" substitution & try to see if it leads to any consequences that look nice. & obey reality.

$$\underbrace{\exp\left\{\frac{ik}{2}\frac{z}{z^2+z_0^2}(x^2+y^2)\right\}}_{\text{spatial phase evolution}}\underbrace{\exp\left\{-\frac{k}{2}\frac{z_0}{z^2+z_0^2}(x^2+y^2)\right\}}_{\text{spatial field amplitude evolution.}}$$

At this point I know this makes no sense :)

MAKE IT LOOK "PRETTIER" & more like a spherical wave & a gaussian beam.

$$\underbrace{\exp\left\{\frac{ik(x^2+y^2)}{2R(z)}\right\}}_{\text{spherical-like}}\underbrace{\exp\left\{\frac{-(x^2+y^2)}{W^2(z)}\right\}}_{\text{gaussian-like.}} \qquad \frac{z_0}{z^2+z_0^2} = \frac{2}{kW^2(z)}$$

NOTE: when arg of exp = -1, the field amplitude is $\frac{1}{e}$ of it's peak at $x=y=0$.

DEF: $\frac{kz_0}{2(z^2+z_0^2)} = \frac{1}{W^2(z)}$ so that when $r^2 = W^2(z)$

field is $\frac{1}{e}$ of its original value.

$W(z) = \frac{1}{e}$ radius of beam

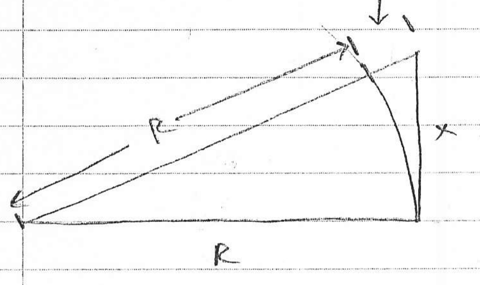
As TKG says in class all the time.

WHAT DOES THIS PHASE TERM REALLY MEAN?
WHY IS z LIKE the RADIUS OF CURVATURE?

RECALL: (GO TO RECALL SECTION)

Consider 1-D example spherical (cylindrical) wave

"EXTRA" optical path length for wavefront to reach the plane of interest @ x w.r.t. the dist R traveled for the axial ray. $(x=0)$



Assume $x \ll R$.

$$OPL(x) = \sqrt{x^2+R^2} - R = R\sqrt{1+\frac{x^2}{R^2}} - R$$

$$= R\underbrace{\left(1+\frac{x^2}{2R^2}+\frac{1}{8}\left(\frac{x^2}{R^2}\right)^2+\cdots\right)}_{\text{ALL TERMS OF TAYLOR EXP}} - R$$

$$= \frac{x^2}{2R} + \frac{x^4}{8R^3} + \cdots\cdots$$

So, when we drop all H.O.T. we get

$OPL(x) \cong \dfrac{x^2}{2R}$ which says that the spherical wavefront must travel $R+\dfrac{x^2}{2R}$ ~~FARTHER than the distance R~~ to reach the plane of interest at a dist x off the optical axis. EXTEND to 3-D

Ray must travel $R+\dfrac{x^2+y^2}{2R}$

R = dist from apparent ~~sphere~~ point source.

"effective radius of curvature at z

so: $$R(z) = \frac{z^2+z_0^2}{z}$$

$$w^2(z) = \frac{(z^2+z_0^2)2}{kz_0} = \frac{2\lambda z_0^2}{2\pi n z_0}\left(1+\left(\frac{z}{z_0}\right)^2\right)$$

look at $w^2(z=0) = w_0$

$$w^2(z) = w_0^2\left(1+\left(\frac{z}{z_0}\right)^2\right) = \frac{1}{e} \text{ width of radius}$$

as a function of z.

From math, we see that NO matter what, w is smallest at $z=0$

where $$w_0^2 = w^2(z=0) = \frac{2\lambda z_0}{2\pi n} = \frac{2z_0}{k}$$

~~PHYSICALLY, w_0 is the $\frac{1}{e}$ radius of the beam at $z=0$~~

OK, what the hell does this all mean?

$w_0 = w(z=0)$ is the $\frac{1}{e}$ radius of the beam at $z=0$. OK, so what? Why is $z=0$ significant?

$$R(z=0) = \frac{z^2+z_0^2}{z} = \infty \quad !!!$$

The radius of curvature is INFINITE at $z=0$ AND the waist is THE SMALLEST IT CAN BE (at least the math says so) at $z=0$

LET'S TALK REALITY - What can we measure? How about w_0?

INTERPRETATION:

$$w^2(z) = w_0^2\left(1+\left(\frac{z}{z_0}\right)^2\right)$$

$$W^2(z) = W_0^2\left[1+\left(\frac{z\lambda}{w_0^2\pi n}\right)^2\right]$$

z_0 is the distance from the ~~longitudinal~~ min-waist position ~~of the waist~~ where the waist has expanded to $\sqrt{2}$ times its original value.

$z_0 = \dfrac{w_0^2\pi n}{\lambda}$: the smaller w_0, the quicker the beam waist expands to $\sqrt{2}$ of its min value.

WHAT ABOUT $R(z)$ the "effective radius of curvature"

$$R(z) = z\left[1+\left(\frac{z_0}{z}\right)^2\right] = \text{Radius of curvature changes as a function of } z$$

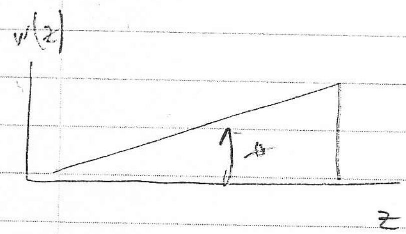
$$R(z) = z\left[1+\left(\frac{\pi n w_0^2}{\lambda_0 z}\right)^2\right]$$

CHECK LIMIT $z \gg z_0$

$$R(z) \simeq z \qquad \left[1 \gg \left(\frac{z_0}{z}\right)^2\right]$$

$$\underset{z\gg z_0}{W^2(z)} = \frac{W_0^2 z^2}{z_0^2} \qquad \left[1 \ll \left(\frac{z}{z_0}\right)^2\right]$$

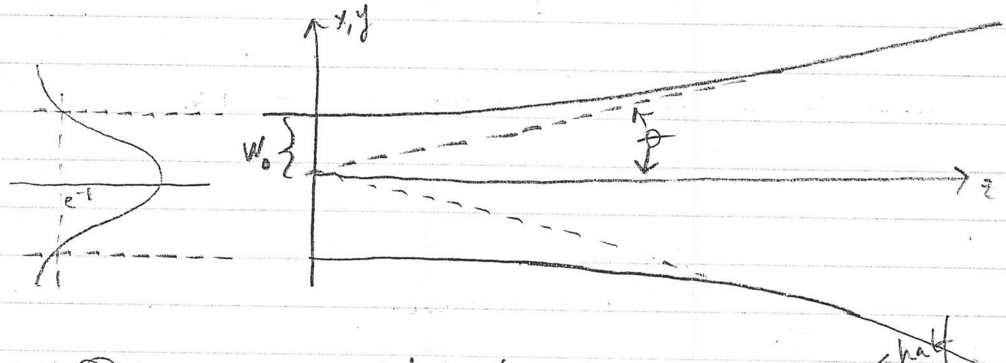
and note: $\dfrac{d[w(z)]}{dz}$ = slope @ $z \simeq \theta$



$$\boxed{\theta = \frac{w_0}{z_0} = \frac{w_0\lambda}{w_0^2\pi n} = \frac{\lambda}{w_0\pi n}}$$

So, at $z=0$ it's as if you're observing the phase from a point source a dist $R(z=0)=\infty$ away!

As z increases, $R(z)$ remain very large but quickly approaches the value z so for $z \gg z_0$ it always looks like the effective point source radiator is at $z=0$!



EX: (A) A commercial He-Ne laser is advertised to have a far field divergence angle of 1 mrad. $\lambda_0 = 632$ nm. What is spot size W_0?

Farfield $z \gg z_0$ half angle $\theta = 1$ mrad.

$$1 \text{ mrad} = \frac{\lambda}{W_0 \pi n} \quad \therefore \quad \boxed{W_0 = \frac{\lambda}{10^{-3}\pi n}}$$

$$\approx \frac{506 \cdot 10^{-9}}{10^{-3}\pi(1)} = \frac{506 \cdot 10^{-6}}{\pi} \approx 150\ \mu m$$

(B) Power = 5 mW what is intensity $\left(\frac{W}{m^2}\right)$ @ waist $(z=0)$? Spot size area $= \pi W_0^2$

$$\therefore \quad I = \frac{\text{POWER}}{\text{AREA}} = \frac{5\text{ mW}}{\pi W_0^2\, m^2} = \frac{5 \cdot 10^{-3}}{\pi (150)^2 (10^{-6})^2} =$$