Diffraction of a Gaussian Alternative!

Consider now a beam that has $E = \frac{1}{2} e^{-\left(\frac{x^2+y^2}{w_0^2}\right)} e^{i\omega t}$ at $z = 0$ (a Gaussian beam - linearly polarized)

Can this propagate in the $z$-direction without boundaries? (Answer: Yes - but it will spread or diffract in the $r = (x^2+y^2)^{1/2}$ direction)

As with the Gaussian pulse, the phasor representation in the $x$ and $y$-directions are

$$\frac{w_0^2}{\sqrt{\pi}} e^{-(k_x x + k_y y)^2/4} e^{i\omega t}$$

As it propagates it becomes

$$\frac{w_0^2}{\sqrt{\pi}} e^{-(k_x x + k_y y)^2/4} e^{i\omega t} e^{-ik_z z}$$

and $k_x x + k_y y + k_z z = k^2 \frac{\omega^2}{c^2}$
Assume $k_x$ and $k_y$ small, then

$$k_y = (k^2 - k_x^2 - k_y^2)^{1/2}$$

$$= k \left( 1 - \frac{k_x^2 + k_y^2}{2k} + \ldots \right)$$

Add up all the phasors

$$e^{i\omega_x - ikz} \int e^{-(k^2 + k_y^2)\omega_x^2/4 + i k_x k_y z}$$

complete the squares in $k_x$ and $k_y$

$$k_x \omega_x^2 + i k_x k_y - i k_x x$$

$$= \left( \omega_x^2 + i \frac{k_x}{2k} \right) k_x - i k_x x = A \left( k_x - \frac{i x}{2A} \right)^2 + \frac{x^4}{4A}$$

Similarly, for $k_y$ terms. Thus (with $g = (k_x - i x/2A)$)

$$e^{-\omega_0^2 t - ikz} \int e^{A \left[ \frac{1}{g} - \frac{x^2}{4A} \right] + \frac{x^4}{4A}}$$

$$\frac{\omega_0^2}{4A^2} e^{i\omega - ikz} \int e^{-\omega^2 g^2 (2\pi)^2}$$

$$= \frac{\omega_0^2}{4A^2} e^{i\omega - ikz} \int e^{\frac{1}{2}\left( \frac{\omega}{\omega_0} \right)^2 - \frac{(\omega^2 + \omega_0^2)}{4A}}$$

$$= \frac{\omega_0^2}{4A^2} e^{i\omega - ikz} \int e^{-\frac{(\omega^2 + \omega_0^2)}{4A}}$$

$$= \frac{1}{\left( 1 + \frac{i x}{2A} \omega_0^2 \right)} e^{i\omega - ikz} \int \frac{1}{\omega^2 + i \frac{x}{\omega_0^2}}$$

$$= \frac{\omega_0^2}{\omega_0^2 + i \frac{x}{\omega_0^2}} - i \frac{x}{\omega_0^2} \frac{1}{\omega_0^2 + i \frac{x}{\omega_0^2}} = \frac{1}{2R(z)} - i k$$

$$\omega(z) = \omega_0 \left( 1 + \frac{3\omega_0}{2z} \right); \quad z_0 = \frac{k}{2} \omega_0^2 = \frac{\omega_0^2}{c} \omega_0^2$$

$R(z) = 2 \left( 1 + \frac{3\omega_0}{2z} \right)$
wavefront (approximate as plane)

Scalar theory (parax approx)

\[ E = \sum_{m} e^{-jkr} \int_{dS} e^{-jkr} \, dS \]

\[ E = \frac{E_0}{r} \int_{dS} e^{-jkr} \, dS \]

\[ (S + r)^2 = x^2 + s^2 \rightarrow 2Sr s = x^2 \quad s \ll r \]

\[ S = x^2 + \frac{k^2}{2} \rightarrow \text{Fresnel} \]

\[ E = \frac{E_0}{r} \int_{dS} e^{-jkr} \, dS \]

\[ = -\frac{E_0}{kr} \int_{x=-\infty}^{x=\infty} e^{-j\frac{k^2}{2}} \, dx \]

\[ = \frac{E_0}{kr} \int_{x=-\infty}^{x=\infty} e^{-j\frac{k^2}{2}} \, dx \]

\[ = \frac{E_0}{kr} \int_{x=-\infty}^{x=\infty} \frac{e^{-j\frac{k^2}{2}}}{\sqrt{2\pi}} \, dx \]

\[ = \frac{E_0}{kr} \left[ \frac{1}{2} + i \frac{1}{2} \right] \]

\[ S(xa) = \int_{x=0}^{x=\infty} \sin \frac{\pi u^2}{2} \, du = \text{Fresnel Sine Int} \]

\[ \int_{x=0}^{x=\infty} \cos \frac{\pi u^2}{2} \, du = \text{Fresnel Cos Int} \]

rest is looking up Cornu Spivel values.
Problem 2

Fourier transform of a lens

\[
\mathcal{F}\{e^{i k \phi}(z, \phi)\} = \int e^{-i k \frac{z}{2} (x'' - x')} dx'
\]

\[
\text{Lenses - curvature change} \quad e^{i k \frac{z}{2} (z - z')} \rightarrow e^{i k \frac{z}{2} \left(\frac{z}{2} - \frac{z'}{2}\right)}
\]

\[
\frac{1}{2} \int \frac{dx'}{\sqrt{z^2 + (z - z')^2}}
\]

\[
\frac{1}{4} \int \frac{dx'}{\sqrt{z^2 + (z - z')^2}}
\]

\[
(x'' - x') - x''(x'' - x' + x'') + (x - x'')^2
\]

\[
= -x'(x'' - x' + x'') + (x - x''')^2
\]

\[
= -2x'(x'' - x''') + (x')^2 + x^2 - 2x'x'' + (x'')^2
\]

\[
= \left[ x'' - (x + x') \right]^2 + (x')^2 + x^2
\]

\[
\text{change integration variable from } x'' \text{ to } x'' - (x + x')
\]

to get Gaussian integral
Problem No. 3

For a solid state material

\[ \text{Re}(\bar{p}) = \sum_{k_1, k_2} \frac{q^2 \mathbf{f}}{k_1 k_2} \sum_{\omega} S(\omega - (E_{k_1} - E_{k_2}))/\omega \text{ in the limit as damping} \to 0 \]

continuous limit. 3d bulk solids

\[ \mathbf{t} = \frac{\omega}{\omega_0} | \mathbf{p}_{21} |^2 \]

\[ 1 + \frac{\mathbf{q}^2}{\hbar} \to \text{oscillator strength} \]

The matrix element:

\[ \mathbf{p}_{21} = \int \psi_2^* \mathbf{e} \mathbf{r} \psi_1 \text{ dV} \]

\[ \psi_2 = \frac{\mathbf{U}_2(r)e^{-i \mathbf{k}_2 \cdot \mathbf{r}}}{\sqrt{V}} \]

\[ \psi_1 = \frac{\mathbf{U}_1(r)e^{-i \mathbf{k}_1 \cdot \mathbf{r}}}{\sqrt{V}} \]

Sum over periodic cells and integrate over unit cell volume

\[ |\mathbf{p}_{21}|^2 = |\text{Min}|^2 \frac{1}{V^3} \sum_{k'} S(\mathbf{K} - \mathbf{K}') (2\pi)^3 \]

\[ |\text{Min}|^2 = \left[ -\frac{i}{\Delta} \int_{\text{cell}} \mathbf{u}_c^*(\mathbf{r}') \mathbf{\hat{e}} \cdot \nabla \mathbf{u}_c(\mathbf{K}'' \mathbf{r}') \text{ d}\mathbf{r}' \right]^2 \]

Note: Have used \( \bar{p}, \mathbf{E} \sim (q \mathbf{r}) \cdot i\omega \mathbf{A} \); \( m \mathbf{\dot{r}} = \bar{p} \)

\( m \frac{\mathbf{q}}{\hbar} (\mathbf{A}) \cdot \hbar \mathbf{V} \)

This generalizes the interaction from dipole to \( \mathbf{H}^1 = \frac{q}{m} \mathbf{A} \cdot \bar{p} \rightarrow \text{all multipoles} \)
Thus

\[ \text{Im} \, P = \frac{q^2}{m^2 \hbar^2} \frac{1}{4 \pi} \int \frac{d^3 k}{(2\pi)^3} \delta \left( \omega - (E_2 - E_1)/\hbar \right) \]

\[ E_2 - E_1 = \frac{h^2}{m} k^2 + E_0 \]

\[ \therefore \quad \alpha(E_2-E_1) = \frac{2\hbar^2}{2m} \int \frac{d^3 k}{(2\pi)^3} \delta \left( \omega - (E_2 - E_1)/\hbar \right) \]

\[ 4\pi k^2 \frac{d^3 k}{(2\pi)^3} = \frac{2\hbar^2}{2m} \int \frac{d^3 k}{(2\pi)^3} \delta \left( \omega - (E_2 - E_1)/\hbar \right) \]

\[ \therefore \quad \text{Im} \, P = \frac{q^2}{m^2 \hbar^2} \frac{1}{4 \pi} \int \frac{d^3 k}{(2\pi)^3} \delta \left( \omega - (E_2 - E_1)/\hbar \right) \]

\[ = \frac{q^2}{m^2 \hbar^2} \frac{1}{4 \pi} \frac{k}{\hbar} \]

In terms of dipole matrix element

\[ = \frac{q^2}{m^2 \hbar^2} \frac{1}{4 \pi} \frac{\omega \omega' \omega'}{E_0} \frac{1}{\hbar} \quad \text{Im} \, \chi \quad \text{E} \]

\[ \omega = \frac{2 \omega \omega'}{\omega} \chi = \frac{2 \omega \omega'}{\omega} \frac{q^2}{m^2 \hbar^2} \chi \quad \text{E} \]

Yavru Quantum Electrodynamics 2nd Ed Eq. 10.2-13

The matrix element can be related to the fluorescent lifetime (8.3-7)

\[ q^2 \chi \omega \omega' = (\omega \omega') \frac{E}{E \omega \omega'} \]

\[ \chi = \frac{1}{m} \frac{2 \pi}{\omega} \left[ \frac{m}{\hbar^2} \right] \frac{1}{\hbar} \left( \frac{E_0 - E}{\hbar} \right) \geq \frac{E}{E \omega \omega'} \]

\[ \geq \frac{1}{m} \frac{2 \pi}{\omega} \left[ \frac{m}{\hbar^2} \right] \frac{1}{\hbar} \left( \frac{E_0 - E}{\hbar} \right) \geq \frac{E}{E \omega \omega'} \]
Classical Radiating Dipole

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\[ W = \frac{\kappa^2}{v} \frac{W^3 p^2 \tau \pm \pi}{32 \pi^2} \]

\[ \frac{dE}{clt} = \frac{hW}{clt} \frac{dN}{clt} = \frac{hW}{clt} \]

\[ \frac{1}{clt} = \frac{1}{\kappa^2} \frac{\sum_{v} \frac{W^3 p^2 \tau \pm \pi}{32 \pi^2} \frac{1}{hW}}{8 \pi} \]

\[ \frac{L}{clt} \frac{C^3 \frac{h}{\pi}}{12 \pi} = \frac{p^3}{3} \rightarrow 12 \frac{p^2}{W^3} \]

\[ \frac{L}{clt} \]
Problem No 4

is certainly true for a pulse after propagation through a fiber, but before compression,

\[ a = \frac{1}{2B} \quad (10.6.35) \]

The bandwidth of the spectrally broadened pulse emerging from the fiber can be estimated by:

\[ (\Delta \omega)^2 = 4t_p^2 \left( \frac{1}{\gamma} + B^2 \right) \text{ or } \Delta \omega = 2Bt_p \quad (10.6.36) \]

if it is assumed that \( B >> \frac{1}{\gamma} \), that is that the chirp is large compared to the initial bandwidth of the pulse divided by initial pulse duration. It now remains to be shown, how a quadratic phase distortion of correct sign can be obtained in a dispersive element. NH 2

10.7 Theory of Compression by Dispersive Elements- Gratings

The compression of pulses with spectra broadened by SPM was first accomplished by Treacy using the grating pair, which has since seen broad application in the optical signal processing area. The anomalous dispersion characteristics arises from a combination of a spectrally dependent path delay and phase shift due to the grating momentum component. It is anomalous because the spectral rate of change of the phase is negative as we will show. The general grating arrangement is illustrated in fig.(9), taken from the initial development given by Tracey [19]. Phase delay arising from different optical paths traveled by the different frequencies is given by:

\[ \Phi(\omega) = \frac{\omega P - 2\pi d}{\Lambda} \tan \theta_r \quad (10.7.1) \]

The second term accounts for a \( 2\pi \) phase shift per ruling in the first order diffraction. This term can be deduced by referring to ?? 7. The incident angle on the first grating is \( \theta_i = \gamma \) and the reflected angle is \( \theta_r = \theta - \gamma \) for wavelength \( \lambda \). For a wavelength shift of \( d\lambda \) the reflection angle is shifted by \( d\theta_r = d\theta \). The path length between the gratings is \( G/\cos(\theta_r) \). The distance between the reflection points on the second grating at \( \lambda \) and \( \lambda + d\lambda \) is then \( dL = (G/\cos(\theta_r)d\theta)/\cos(\theta_r) \). The phase shift \( d\phi \) is thus \( -dL^2/2d \). Thus

\[ \frac{d\phi}{d\theta_r} = \frac{2\pi d}{G/\cos(\theta_r)^2} \quad (10.7.2) \]

Integrating this gives the required extra phase term above. The optical path \( P \) can be expressed as,

\[ P = AD + DE = \frac{d}{\cos \theta_r} \left( \cos(\theta_r + \theta_i) + 1 \right) \quad \xi = \xi \quad (10.7.3) \]

The grating formula reveals that \( \theta_r \) and \( \theta_i \) are related through the following equation:

\[ \sin \theta_r = \sin \theta_i + \frac{n\lambda}{\Lambda} \quad (10.7.4) \]

where \( n \) is the order of the grating reflection and \( \Lambda \) is the spatial grating period, but here \( n = -1 \).

Since we are interested in dispersion parameter \( a = -\frac{\lambda^2}{2\Lambda^2} \), taking second derivative of \( \Phi(\omega) \) in eq.(2.2.1) yields:

\[ \frac{\partial^2 \Phi}{\partial \omega^2} = \frac{P}{c} + \frac{\omega}{c} \frac{\partial P}{\partial \omega} - 2\pi \frac{d}{\Lambda} \frac{\partial}{\partial \omega} \tan \theta_r = \frac{P}{c} \quad (10.7.5) \]
Figure 10.7.1: The Teray Grating Pair

\[ \frac{\partial^2 \Phi}{\partial \omega^2} = \frac{1}{c} \frac{\partial P}{\partial \omega} = \frac{d}{c} \left( \frac{\lambda}{\cos^2 \theta, \Lambda} \right) \frac{\partial \theta_r}{\partial \omega} \]  

(10.7.6)

taking the derivative of the grating equation (eq 2.2.2),

\[ \frac{\partial \theta_r}{\partial \omega} = \frac{\lambda}{\omega \Lambda \cos \theta_r} \]  

(10.7.7)

plugging the above result into eq.(2.2.4), yields an expression for the parameter \( a \):

\[ a = \frac{d \lambda^2}{c \omega \Lambda^2 (\cos^2 \theta)^{\frac{3}{2}}} = \frac{4\pi^2 dc}{\omega^3 \Lambda^2} \left( 1 - \left( \sin \theta_i + \frac{\lambda}{\Lambda} \right)^2 \right)^{-\frac{3}{2}} \]  

(10.7.8)

A similar expression for the compression parameter \( a \) in a prism pair is developed next [20], [21]. A desired prism pair arrangement is shown in fig.(10). Again, the equivalent optical path \( P \) is given by an equation similar to the one for a grating pair,

\[ P = \Phi \frac{\omega}{c} = \eta_2 l \cos \theta \]  

(10.7.9)

There is no need for a term accounting for a phase shift unlike the previous case. Parameter \( a \) is obtained in a usual manner, by taking the second derivative of the phase:

\[ a = \frac{\partial^2 \Phi}{\partial \omega^2} = \frac{\lambda^3}{2 \pi c^3} \frac{\partial^2 P}{\partial \lambda^2} \]  

(10.7.10)

Some further manipulation yields a following expression for \( \frac{\partial^2 P}{\partial \lambda^2} \):

\[ \frac{\partial^2 P}{\partial \lambda^2} = -l \left( \cosh \left( \frac{\partial \theta}{\partial \lambda} \right)^2 + \sinh \left( \frac{\partial^2 \theta}{\partial \lambda^2} \right) \right) \]  

(10.7.11)

\( a \) must cancel the opposite chirp on the pulse to eliminate phase modulation.