Notes on ABCD Matrix Methods February 22, 2009

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ABCD Matrix Methods for Analyzing Optical Systems

When you want to examine a complicated optical system with many components (such as an eye, a telescope, binoculars, microscope, or some fancy gadget you design), it can be cumbersome to calculates object and image distances for each component. When you are dealing with the paraxial regime and one axis, you can use a straightforward matrix method to simplify your calculation.

In this method, we'll consider what happens to a ray a distance r from the optic axis, traveling in a direction θ as measured from the positive direction, as illustrated in figure 1

$$\begin{bmatrix} r \\ \theta \end{bmatrix}$$

First, let's consider what happens to r and θ when the ray is propagating a distance l through a

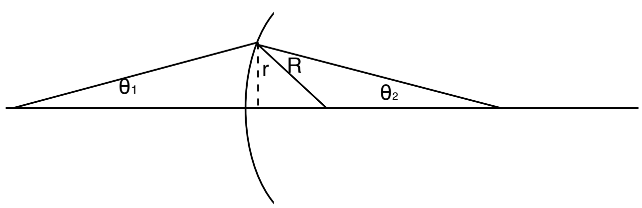

Figure 1:

medium of refractive index n. If the medium is homogeneous, it doesn't ever get bent, so θ will not change. If θ is nonzero, r will change by $|\tan(\theta)$. When we are dealing with the paraxial regime, we can say that r changes by $l\theta$, and then we have a transfer matrix of a ray propagating through a homogeneous medium a distance l:

$$\begin{bmatrix} r_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ \theta_1 \end{bmatrix}$$

Now let's consider refraction at a curved surface of radius R. R¿0 as drawn in figure **??** because the center of the arc is to the left of the lens position. Let α be the angle between the optic axis and the radius of curvature of the survace where the ray strikes the surface. Then if the ray is at position r then $\sin(\alpha) = \frac{r}{R}$. The angle of incidence between the surface and the ray is $\phi_1 = \alpha + \theta$.

Now we use snell's law:
and when we go through all this math we see that in the paraxial region,

$$\theta_2 = \frac{n_1}{n_2}\theta_1 + \frac{n_1 - n_2}{n_2}\frac{r}{R}$$

And since we are just looking at refraction, r doesn't change. So we get that the transfer matrix for a curved interface is

$$\begin{bmatrix} r_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{n_1-n_2}{Rn_2} & \frac{n_1}{n_2} \end{bmatrix} \begin{bmatrix} r_1 \\ \theta_1 \end{bmatrix}$$

If the radius of curvature is negative, the same expression will hold, except we'll use a negative value for R. So if we want to put two refractive surfaces together to form a thin lens, we can do that easily. Here n_1 is the refractive index of the surrounding medium, and n_2 is the refractive index of the glass.:

$$\begin{bmatrix} r_3 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{n_2-n_1}{R_2n_1} & \frac{n_2}{n_1} \end{bmatrix} \begin{bmatrix} r_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{n_2-n_1}{R_2n_1} & \frac{n_2}{n_1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{n_1-n_2}{R_1n_2} & \frac{n_1}{n_2} \end{bmatrix} \begin{bmatrix} r_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{n_2-n_1}{R_2n_1} + \frac{n_1-n_2}{R_1n_2}\frac{n_2}{n_1} & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ \theta_1 \end{bmatrix}$$

Simplifying the nontrivial element of the final matrix and recognizing the lensmaker's equation,

$$\frac{n_2 - n_1}{R_2n_1} + \frac{n_1 - n_2}{R_1n_1} = \frac{n_2 - n_1}{n_1}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) = -\frac{1}{f_{\text{lens}}}$$

So for a Gaussian thin lens, we have

$$\begin{bmatrix} r_3 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ \theta_1 \end{bmatrix}$$

Notice that if the rays pass into from a medium of refractive index n_1 a medium of refractive index n_2, you can model it as an interface with infinite radius, and the matrix for that would be

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix}$$