EE 119 Section February 22, 2009

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Diffraction Limits the Resolution of Optical Systems

Any image captured through a finite aperture contains some fraction of the information of the entire image, because a finite aperture only captures a finite fraction of the rays leaving an image. The intensity of the light from a point source exiting an aperture of radius a at angle θ is given by Hecht 10.56, and also in the notes, to be

$$I(q,R) = I(0)\frac{2J_1(kaq/R)}{kaq/R}$$

Here k is the wavenumber of the light, R is the distance from the aperture to the screen, and q is the position on the screen. These are related to the exit angle by $\sin(\theta) = q/R$, so an equivalent formulation is

$$I(\theta) = I(0)\frac{2J_1(ka\sin(\theta))}{ka\sin(\theta)}$$

Which expression you choose to use depends on what kind of optical instrument you're analyzing. The Bessel Function $J_1(x)$ is a function that is similar to a a sine function, except that it decreases in amplitude with increasing values of its argument-see p. 469 of Hecht for a picture. Its first zero is located at x=3.83, which is something you have to solve for numerically. From that, we get the relationship that the radius of an image is

$$q = 1.22\frac{R\lambda}{2a}$$

Or, using angles

$$\sin(\theta) = 1.22\frac{\lambda}{2a} = 1.22\frac{\lambda}{D}$$

where D=2a is the diameter. We can use this to compute the diffraction limit of an optical system such as our eye

Diffraction limit of the eye

In average light intensity, the pupil of the eye is 4 mm in diameter, and the average wavelength is 500 nm. What is the resolution imposed by the eye?

The Eye

1. If the index of refraction of the eye were 1 (the same as air), and we modeled the eye as a single lens, what would be the length of the eye? We know that the power of a normal eye is 58.6 Diopters.

$$\frac{1}{d_2} = 58.6 + \frac{1}{\infty}$$
$$d_2 = 0.017 = 1.17\text{cm}$$

2. The refractive index of the humors in the eye is 1.336. Using this value, what is a more accurate estimate for the length of the eye?

3. The separation of cones in the fovea is 1'=1/60°.=0.3 milliradians. The refractive index of the humors in the eye is 1.336. What is the distance between cones in the fovea?

4. I am nearsighted, and the power of my glasses is -3 diopters. What Is the farthest distance at which I can clearly focus on an object?

ABCD Matrix Methods for Analyzing Optical Systems

When you want to examine a complicated optical system with many components (such as an eye, a telescope, binoculars, microscope, or some fancy gadget you design), it can be cumbersome to calculates object and image distances for each component. When you are dealing with the paraxial regime and one axis, you can use a straightforward matrix method to simplify your calculation.

In this method, we'll consider what happens to a ray a distance r from the optic axis, traveling in a direction θ as measured from the positive direction, as illustrated in figure **??**

$$\begin{bmatrix} r \\ \theta \end{bmatrix}$$

First, let's consider what happens to r and θ when the ray is propagating a distance l through a

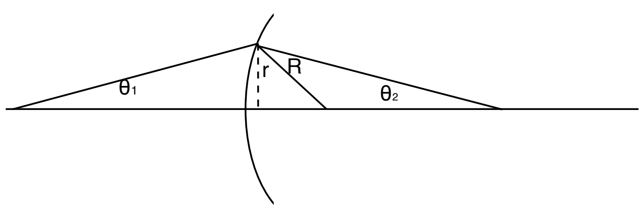

Figure 1: diagram for problem **??**.

medium of refractive index n. If the medium is homogeneous, it doesn't ever get bent, so θ will not change. If θ is nonzero, r will change by $|\tan(\theta)$. When we are dealing with the paraxial regime, we can say that r changes by $l\theta$, and then we have a transfer matrix of a ray propagating through a homogeneous medium a distance l:

$$\begin{bmatrix} r_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ \theta_1 \end{bmatrix}$$

Now let's consider refraction at a curved surface of radius R. R¿0 as drawn in figure **??** because the center of the arc is to the left of the lens position. Let α be the angle between the optic axis and the radius of curvature of the survace where the ray strikes the surface. Then if the ray is at position r then $\sin(\alpha) = \frac{r}{R}$. The angle of incidence between the surface and the ray is $\phi_1 = \alpha + \theta$.

Now we use snell's law:

and when we go through all this math we see that in the paraxial region,

$$\theta_2 = \frac{n_1}{n_2}\theta_1 + \frac{n_1 - n_2}{n_2}\frac{r}{R}$$

And since we are just looking at refraction, r doesn't change. So we get that the transfer matrix for a curved interface is

$$\begin{bmatrix} r_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{n_1-n_2}{Rn_2} & \frac{n_1}{n_2} \end{bmatrix} \begin{bmatrix} r_1 \\ \theta_1 \end{bmatrix}$$

If the radius of curvature is negative, the same expression will hold, except we'll use a negative value for R. So if we want to put two refractive surfaces together to form a thin lens, we can do that easily. Here n_1 is the refractive index of the surrounding medium, and n_2 is the refractive index of the glass.:

$$\begin{bmatrix} r_3 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{n_2-n_1}{R_2n_1} & \frac{n_2}{n_1} \end{bmatrix} \begin{bmatrix} r_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{n_2-n_1}{R_2n_1} & \frac{n_2}{n_1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{n_1-n_2}{R_1n_2} & \frac{n_1}{n_2} \end{bmatrix} \begin{bmatrix} r_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{n_2-n_1}{R_2n_1} + \frac{n_1-n_2}{R_1n_2}\frac{n_2}{n_1} & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ \theta_1 \end{bmatrix}$$

Simplifying the nontrivial element of the final matrix and recognizing the lensmaker's equation,

$$\frac{n_2 - n_1}{R_2n_1} + \frac{n_1 - n_2}{R_1n_1} = \frac{n_2 - n_1}{n_1}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) = -\frac{1}{f_{\text{lens}}}$$

So for a Gaussian thin lens, we have

$$\begin{bmatrix} r_3 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} + \frac{n_1-n_2}{R_1n_1} & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ \theta_1 \end{bmatrix}$$

Notice that if the rays pass into from a medium of refractive index n_1 a medium of refractive index n_2, you can model it as an interface with infinite radius, and the matrix for that would be

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix}$$

Example

In problem set 3, you had the following problem:

A compound lens is composed of two thin lenses separated by 10 cm. The first of these has a focal length of +20cm, and the second a focal length of 30cm. Determine the focal length of the combination and locate the corresponding principal points.

You can solve this problem using matrices instead of using the formula for compound lenses. We have three matrices:

$$\begin{bmatrix} r_4 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{30} & 1 \end{bmatrix} \begin{bmatrix} r_3 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{30} & 1 \end{bmatrix} \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{30} & 1 \end{bmatrix} \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{20} & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ \theta_1 \end{bmatrix}$$

$$\begin{bmatrix} r_4 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 1/2 & 10 \\ -2/30 & 2/3 \end{bmatrix} \begin{bmatrix} r_1 \\ \theta_1 \end{bmatrix}$$

The effective focal point of the system is the distance d needed to propagate r_4 and θ_4 so that $r_4 = 0$ for all values of θ_4, when r_4 and θ_4 are obtained from $\theta_1 = 0$:

$$\begin{bmatrix} r_5 \\ \theta_5 \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 10 \\ -2/30 & 2/3 \end{bmatrix} \begin{bmatrix} r_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_1/2 \\ -2r_1/30 \end{bmatrix} = \begin{bmatrix} r_1/2 - 2dr_1/30 \\ -2r_1/30 \end{bmatrix}$$

So we see that $r_1 = 0$ when $r_1/2 = 2dr_1/30$, which is when $d = 7.5$. so the effective focal length of the compound lens system is 7.5 cm, which is what you shoult have gotten for the problem by tracing out objects and images.

This method was a lot easier than finding image and object distances. It would be cumbersome to multiply all those matrices by each other, but that's what computers can be used for.