Chapter 6
SCALAR DIFFRACTION THEORY

[Reading assignment: Hect 10.2.4-10.2.6,10.2.8, 11.3.3]

Scalar Electromagnetic theory:

monochromatic wave

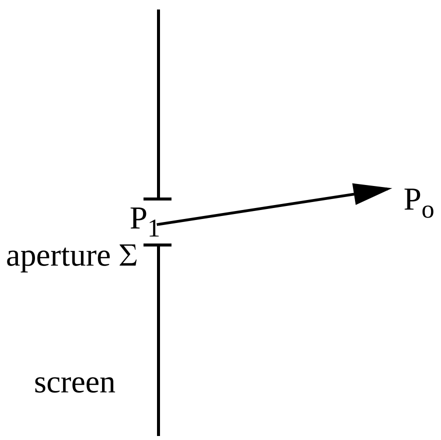
P : position *t* : time $\omega = 2\pi\nu$: optical frequency

u(P, t) represents the *E* or *H* field strength for a particular transverse polarization component

U(P) : represents the complex field amplitude

$U(P) = u(P)e^{-j\phi(P)}$ $u(P)$: real

Diffraction:



Approximations:

1. We impose the boundary condition on *U*, that *U* = 0 on the screen.
2. The field in the aperture Σ is not affected by the presence of the screen.

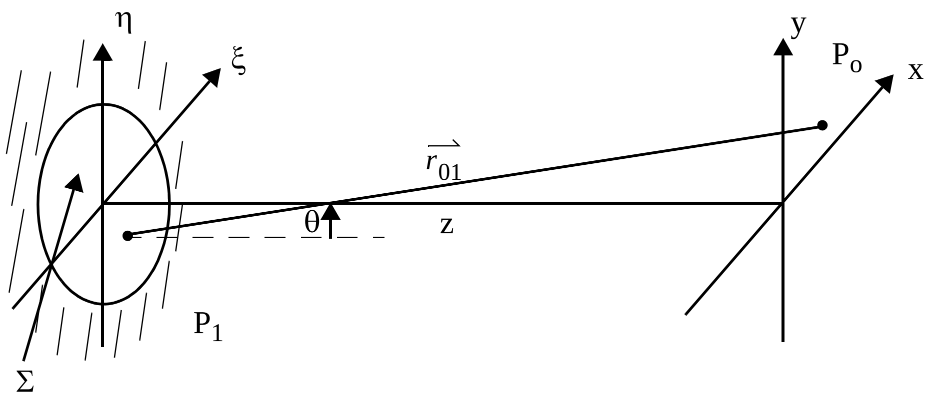
$$U(P_o) = \frac{1}{j\lambda}\iint_{\Sigma} U(P_1)\underbrace{\frac{\exp(jkr_{01})}{r_{01}}}_{\text{expanding spherical}}ds$$

$[r_{01} \gg \lambda]$

This equation expresses the Huygens-Fresnel principle: The observed field is expressed as a superposition of point sources in the aperture, with a weighting factor ______.

Fresnel approximation

Huygens-Fresnel integral in rectangular coordinates:



The Fresnel approximation involves setting: $r_{01} \cong z$ in the denominator, and

$$r_{01} \cong z\left[1 + \frac{1}{2}\frac{(x-\xi)^2}{z^2} + \frac{1}{2}\frac{(y-\eta)^2}{z^2}\right] \quad \text{in exponent}$$

This is equivalent to the paraxial approximation in ray optics.

$$U(x,y) = \frac{\exp(jkz)}{j\lambda z}\int\int_{-\infty}^{\infty} d\xi d\eta\, U(\xi,\eta)\exp\left\{\frac{jk}{2z}[(x-\xi)^2 + (y-\eta)^2]\right\} \qquad \text{(A)}$$

Let's examine the validity of the Fresnel approximation in the Fresnel integral. The next higher order term in exponent must be small compared to 1. So the valid range of the Fresnel approximation is:

$$z^3 \gg \frac{\pi}{4\lambda}[(x-\xi)^2 + (y-\eta)^2]^2_{max}$$

For field sizes of 1 cm, $\lambda = 0.5\mu m$, we find $z \gg 25$ cm.

Actually we should look at the effect on the total integral. Upon closer analysis, it is found that the Fresnel approximation holds for a much closer z. This is referred to as the "near-field region".

Farther out in z, we can approximate the quadratic phase as flat

$$z \gg \frac{k(\xi^2 + \eta^2)_{max}}{2}$$

This region is referred to as the "far-field" or Fraunhofer region.

$$U(x, y) = \frac{e^{jkz} e^{j\frac{k}{2z}(x^2 + y^2)}}{j\lambda z} \underbrace{\iint d\xi d\eta U(\xi, \eta) \exp\left[-j\frac{2\pi}{\lambda z}(x\xi + y\eta)\right]}_{F\{U(\xi, \eta)\}\Big|_{f_x = \frac{x}{\lambda z}, f_y = \frac{y}{\lambda z}}}$$

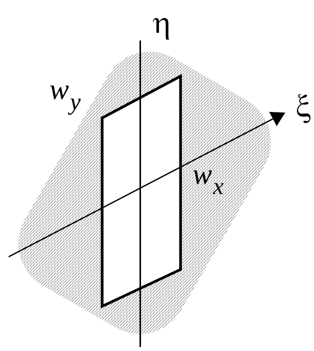
Now this is exactly the Fourier transform of the aperture distribution with

The Fraunhofer region is farther out. For the field size of 1 cm, and $\lambda = 0.5\mu m$, we find the valid range of $z \gg 150$ meters!

Again, examining the full integral, Fraunhofer is actually accurate and usable to much closer distances.

Examples

A rectangular aperture, illuminated by a normally incident plane wave:



$$t_A = rect\left(\frac{\xi}{2w_x}\right) rect\left(\frac{\eta}{2w_y}\right)$$

With plane wave illumination, we have: $U(\xi, \eta) = t_A(\xi, \eta)$

$$\therefore \qquad U(x, y, z) = \frac{e^{jkz} e^{j\frac{k}{2z}(x^2 + y^2)}}{j\lambda z} F[U]\Big|_{\substack{f_x = \frac{x}{\lambda z} \\ f_y = \frac{y}{\lambda z}}}$$

$$= \frac{e^{jk\left[z + \frac{x^2+y^2}{z}\right]}}{j\lambda z} A\,\mathrm{sinc}\left(\frac{2w_x x}{\lambda z}\right)\,\mathrm{sinc}\left(\frac{2w_y y}{\lambda z}\right)$$

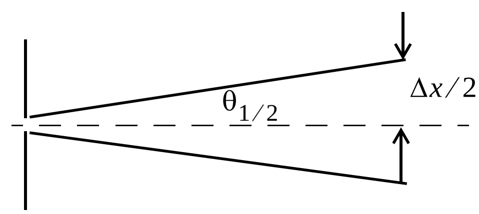
$$A \equiv 4w_x w_y$$

Recall . The observable is intensity $I = |U|^2$.

$$I = \frac{A^2}{\lambda^2 z^2}\mathrm{sinc}^2\left(\frac{2w_x x}{\lambda z}\right)\mathrm{sinc}^2\left(\frac{2w_y y}{\lambda z}\right)$$

The width of the central lobe of the diffraction pattern is

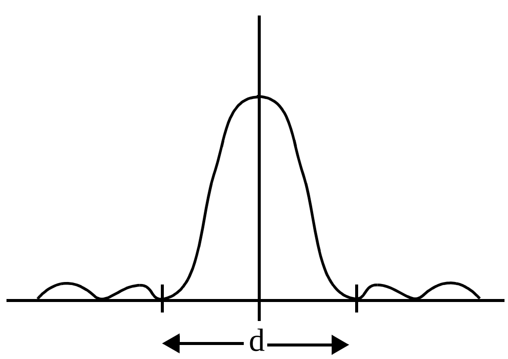
The diffraction half angle $\theta_{1/2} \cong \frac{\Delta x}{2} = \frac{\lambda}{2w_x}$

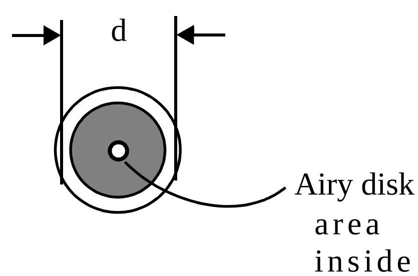


For a circular aperture with radius w : $t_A = circ\left(\frac{q}{w}\right)$ $\quad q^2 \equiv \xi^2 + \eta^2$ radial coordinates

In circular coordinates, we use the Fourier - Bessel transform: $B\{U(q)\}$ gives immediately:

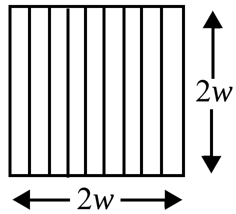
$$I(r) = \left(\frac{A}{\lambda z}\right)^2\left[2\frac{J_1(kwr/z)}{kw(r/z)}\right]^2 \quad \text{"Airy pattern"}$$





diameter of Airy disk

Diffraction grating (transmission)

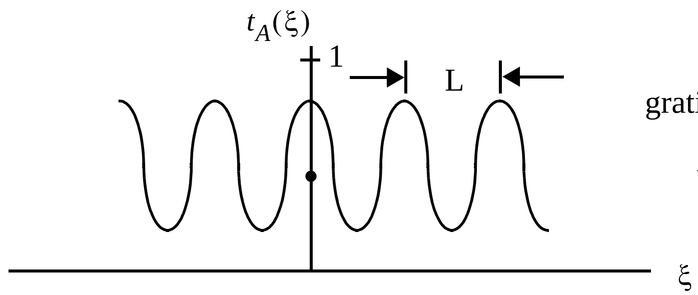


sinusoidal amplitude

$$t_A(\xi,\eta) = \left[\frac{1}{2} + \frac{m}{2}\cos(2\pi f_0 \xi)\right]\mathrm{rect}\left(\frac{\xi}{2w}\right)\mathrm{rect}\left(\frac{\eta}{2w}\right)$$

m: peak to peak amplitude change $\quad 0 \le m \le 1$

f_0: grating spatial frequency

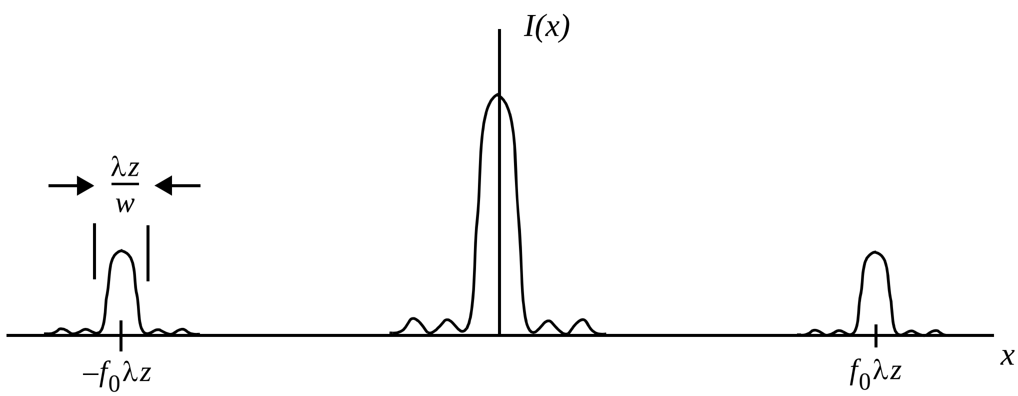


grating period

$$L = \frac{1}{f_0}$$

By convolution, the diffracted amplitude is

$$F\left[\frac{1}{2} + \frac{m}{2}\cos(2\pi f_0 \xi)\right] \otimes F\left[rect\frac{\xi}{2w} rect\frac{\eta}{2w}\right]$$



$$I(x,y) \cong \left(\frac{A}{2\lambda z}\right)^2 \mathrm{sinc}^2\frac{2wy}{\lambda z}\left\{\mathrm{sinc}^2\left(\frac{2wx}{\lambda z}\right) + \frac{m^2}{4}\mathrm{sinc}^2\left[\frac{2w}{\lambda z}(x + f_0\lambda z)\right] + \frac{m^2}{4}\mathrm{sinc}^2\left[\frac{2w}{\lambda z}(x - f_0\lambda z)\right]\right\}$$

We have neglected interference terms between orders.

Compared to the square aperture, which has the central peak with intensity I_o, we now have:

The "resolving power" of the grating

DIFFRACTION THEORY OF A LENS

We have previously seen that light passing through a lens experiences a phase delay given by:

$$\varphi(\xi, \eta) = \exp\left[-jk(n-1)\left(\frac{\xi^2+\eta^2}{2}\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\right] \quad \text{(neglecting the constant phase)}$$

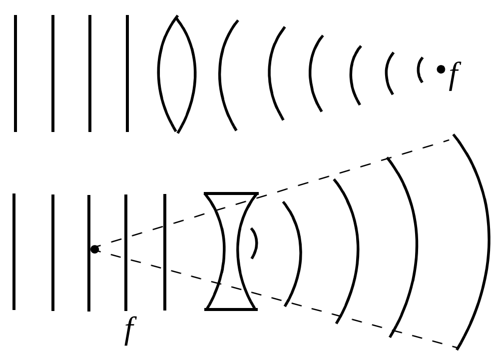
The focal length, f is given by:

The "lens makers formula"

The transmission function is now:

This is the paraxial approximation to the spherical phase

Note: the incident plane-wave is converted to a spherical wave converging to a point at f behind the lens (f positive) or diverging from the point at f in front of lens (f negative).



<u>Diffraction from the lens pupil</u>

Suppose the lens is illuminated by a plane wave, amplitude A. The lens "pupil function" is $P(\xi, \eta)$.

The full effect of the lens is $U_l'(\xi, \eta) = \varphi(\xi, \eta)P(\xi, \eta)$

$$U_l'(\xi, \eta) = P(\xi, \eta)\exp\left[-j\frac{k}{2f}(\xi^2 + \eta^2)\right]$$

We now use the Fresnel formula to find the amplitude at the “back focal plane” $z = f$

$$U_f(x, y) = \frac{\exp\left[j\frac{k}{2f}(x^2 + y^2)\right]}{j\lambda f} \times e^{jkf}\int_{-\infty}^{\infty}\int d\xi d\eta U_l'(\xi, \eta)\exp\left[j\frac{k}{2f}(\xi^2 + \eta^2)\right]\exp\left[-j\frac{2\pi}{\lambda f}(\xi x + \eta y)\right]$$

The phase terms that are quadratic in $\xi^2 + \eta^2$ cancel each other.

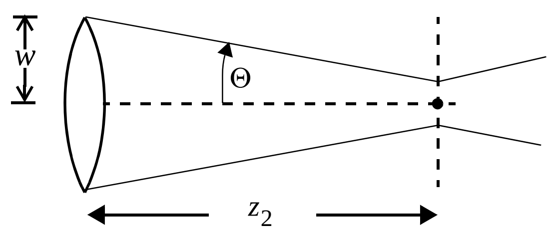
$$U_f(x, y) = \frac{\exp\left[j\frac{k}{2f}(x^2 + y^2)\right]}{j\lambda f} e^{jkf}\int_{-\infty}^{\infty}\int d\xi d\eta P(\xi, \eta)\exp\left[-j\frac{2\pi}{\lambda f}(\xi x + \eta y)\right] \quad \text{(B)}$$

This is precisely the Fraunhofer diffraction pattern of P ! Note that a large z criterion *does not* apply here.

The focal plane amplitude distribution is a Fourier transform of the lens pupil function $P(\xi,\eta)$, multiplied by a quadratic phase term. However, the intensity distribution is exactly

$$I_f(x, y) = \frac{A^2}{\lambda^2 f^2}\left|F[P(\xi, \eta)]\right|^2 \qquad f_x = \frac{x}{\lambda f} \qquad f_y = \frac{y}{\lambda f}$$

Example: a circular lens, with radius w



$$P = \mathrm{circ}\left(\frac{q}{w}\right) \qquad (q^2 = \xi^2 + \eta^2)$$

$$\text{let} \quad h(r) = F[P(\lambda z_2 q)] = F\left[\mathrm{circ}\left(\frac{\lambda z_2 q}{w}\right)\right] \qquad (r^2 = x^2 + y^2)$$

$$= \frac{A}{\lambda z_2}\left[2\frac{J_1(2\pi wr/\lambda z_2)}{2\pi wr/\lambda z_2}\right]$$

$$\left|h(r)\right|^2 = \frac{A^2}{\lambda^2 z_2^2}\left[2\frac{J_1(2\pi wr/\lambda z_2)}{2\pi wr/\lambda z_2}\right]^2$$

The spot diameter is

$$= 1.22\frac{\lambda}{\Theta} \quad \text{paraxial approximation}$$

The "Rayleigh" resolution of the lens is $d/2 = 0.66\lambda/\Theta$.

For a large Θ,