EE 119 Homework 11 Solution

Professor: Jeff Bokor TA: Xi Luo

1. *General diffraction questions*

(a) How many wavelengths wide must a single slit be if the first Fraunhofer diffraction minimum occurs at an angular distance of 30 degrees from the optic axis?

Solution: The Fraunhofer diffraction minimum occurs when

$$\sin c(\frac{2wx}{\lambda z}) = 0 \quad \text{or} \quad \sin(\frac{2wx}{\lambda z}\pi) = 0\,;$$

And the first minimum corresponds to when $\frac{2wx}{\lambda z}\pi = \pi$

$$w = \frac{\lambda z}{2x} = \frac{\lambda}{2\tan\theta} = \frac{\sqrt{3}}{2}\lambda \approx 0.866\lambda\,.$$

So the full-width of this single slit has to be 2w, which is around 1.732λ.

(b) Lycopodium seeds, which are of spherical shape and nearly uniform size, are dusted on a glass plate. If with parallel light of 640nm wavelength the angular radius of the first diffraction maximum is 2 degrees, estimate the size of the seed.

Solution: The angular radius of the first diffraction maximum of a spherical shape object is

$$\theta_{\frac{1}{2}} = 0.61\frac{\lambda}{w} = 2^\circ = 0.035rad\,,$$

in which w is the radius of the seed.

Therefore, the size of the seed is

$$w = 0.61\frac{\lambda}{\theta_{\frac{1}{2}}} = 11.15\mu m \quad \text{in radius.}$$

(c) If the headlights of an approaching car are 122cm apart, what is the maximum distance at which the eye can resolve them? Assume the pupils of 4mm in diameter and light of 500nm wavelength. Consider diffraction limit only.

Solution: The diffraction-limited angular resolution of the pupils is

$$\theta = 1.22\frac{\lambda}{D_{pupil}} < \frac{122cm}{z}$$

in which z is the distance at which the eye can resolve the two headlights.

So the maximum distance is $z_{\max} = \frac{122cm}{1.22}\cdot\frac{\lambda}{D_{pupil}} = 100cm\cdot\frac{4mm}{500nm} = 8km\,.$

2. ***Diffraction grating.*** A molecule sometimes emits light at 600nm and sometimes emits light at 650nm. You want to determine the relative intensity of emission at these two wavelengths, so you decide to split the light with a diffraction grating and direct the two first-order diffracted beams of different colored light onto two different photodetectors which are placed 10cm away from the grating. You want to separate the centers of the photodiodes by 2cm. The active area of the photodiode is 0.5mm × 0.5mm, and to maximize efficiency you want all of the light at the two wavelengths to hit the active area. Please design the first-order diffraction grating you will use.

   Solution: You want the spacing between the first order diffracted beam at 650nm to be 2cm away from the first order diffracted beam at 600nm. This means that

   $$x_{650} - x_{600} = 0.02\text{m}$$

   You know that the first diffracted beam occurs at $x \pm f_0\lambda z$ where f_0 is the grating frequency and z is the distance between the photodiode and the grating. So our restriction means that

   $$f_0 z(650 - 600) \times 10^{-9} \geq 2 \times 10^{-2}\text{m}$$
   $$f_0 z \geq 4 \times 10^{5}$$

   Since we plan to put the photodetectors 10cm away from the grating, z = 0.1m, the frequency of the groove is then

   $$f_0 \geq 4 \times 10^{6}.$$

   We have another restriction, which is that the width of a peak must be less that 0.5 mm=5 × 10^{-3}m to fit into the size of a detector. This means

   $$\frac{\lambda z}{w} \leq 5 \times 10^{-4} m$$

   so for the larger wavelength, 650 nm,

   $$w \geq \frac{10 \times 6.5}{5} \times 10^{-3} = 1.3cm\ .$$

   Therefore, our grating must be at least 1.3cm wide and have at least 4,000,000 grooves per meter.

3. ***Interference*** Sketch the interference pattern produced in the x-y plane by two plane waves, where for wave 1, the wavevector is $\mathbf{k}_1 = (2\pi/\lambda)\ (\mathbf{x} + \mathbf{y} + \mathbf{z})$, and for wave 2, the wavevector is $\mathbf{k}_2 = (2\pi/\lambda)\mathbf{z}$. Take λ = 500 nm. Quantitatively label the dimensions on your sketch.

   Solution: The electric field of each wave at point r is is $E(\vec{r}) = E_0 \cos(\vec{k} \cdot \vec{r})$, so the total electric field is

   $$E_{total} = E_1 \cos(\frac{2\pi}{\lambda}(\vec{x} + \vec{y} + \vec{z}) \cdot (x\vec{x} + y\vec{y} + z\vec{z})) + E_2 \cos(\frac{2\pi}{\lambda}\vec{z} \cdot (x\vec{x} + y\vec{y} + z\vec{z}))$$
   $$= E_1 \cos(\frac{2\pi}{\lambda}(x + y + z)) + E_2 \cos(\frac{2\pi}{\lambda}z)$$

   The intensity will be the square of the electric field:

$$I = |E_{total}|^2 = E_1^2 \cos^2(\frac{2\pi}{\lambda}(x+y+z)) + E_2^2 \cos^2(\frac{2\pi}{\lambda}z) + 2E_1E_2 \cos(\frac{2\pi}{\lambda}(x+y+z))\cos(\frac{2\pi}{\lambda}z)$$

The interference term is

$$I_{int} = I_0 \cos(\frac{2\pi}{\lambda}(x+y+z))\cos(\frac{2\pi}{\lambda}z) = I_0[\cos(\frac{2\pi}{\lambda}(x+y+z)+\frac{2\pi}{\lambda}z) + \cos(\frac{2\pi}{\lambda}(x+y+z)-\frac{2\pi}{\lambda}z)]$$

$$I_{int} = I_0[\cos(\frac{2\pi}{\lambda}(x+y+2z)) + \cos(\frac{2\pi}{\lambda}(x+y))]$$

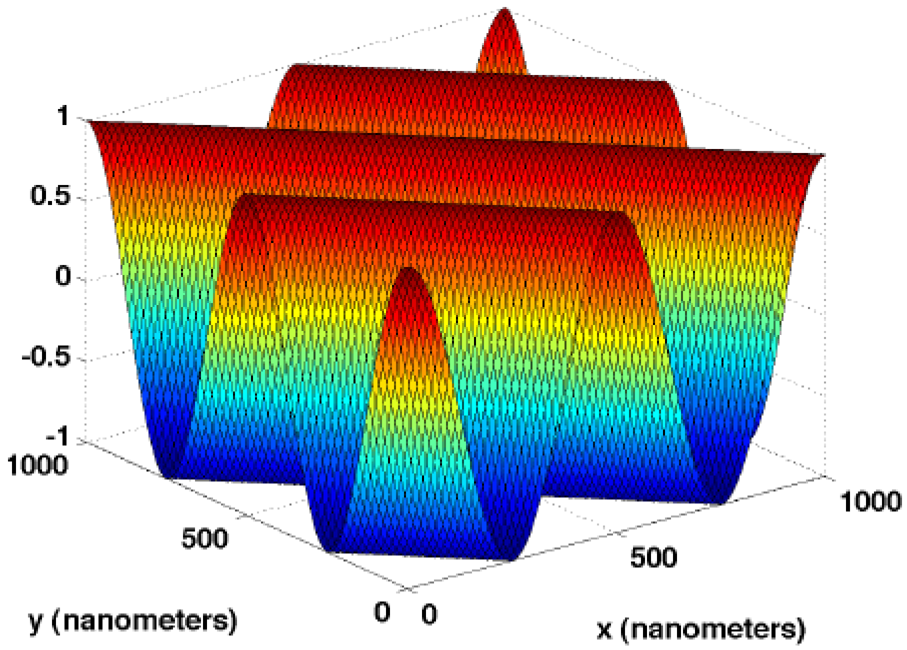
In the x-y plane, we can ignore the dependence on z (and pick a constant value for z, say z=0), and the variation of intensity will be $\cos(\frac{2\pi}{\lambda}(x+y))$.

When $x + y = m\lambda = m \cdot 500nm$, the total intensity is maximum;

when $x + y = m\lambda + \frac{\lambda}{2} = (m \cdot 500 + 250)nm$, the total intensity reaches minimum.

(m is an integer).

The interference pattern in the x-y plane is then shown in the following figure.



4. ***Young's double slit Experiment.*** Young's double slit experiment is described in Fig.1. Answer following questions.
   (a) Plot the intensity pattern at the observing plane, P_o. Label x and y-axis clearly.
   (b) Light passes through two slits separated by a distance d=0.8mm, and the observing plane is 1.6m away from the two slits. If the distance between the two consecutive maxima is 5mm, what is the wavelength of the light?
   (c) When one of the slits is covered by a film of transparent material, the zeroth order is seen to shift by 2.2 fringes. If the refractive index of the transparent material is 1.4, how thick is the film?

(d) The two slits are illuminated by light containing two wavelengths, 450nm and 600nm. What is the least order at which a maximum of one wavelength will fall exactly on a minimum of the other?

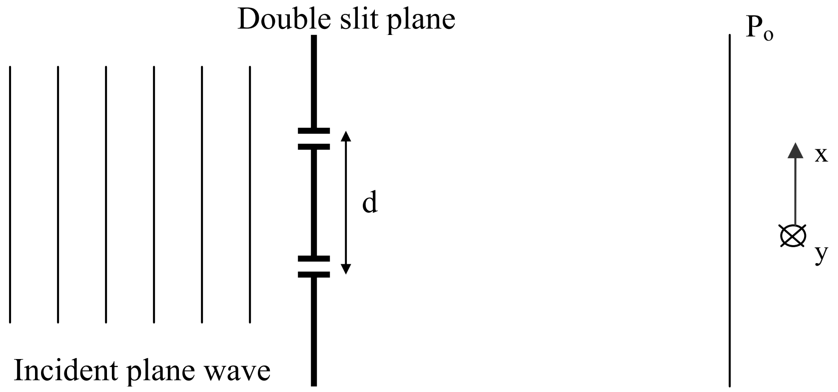


Fig.1

Solution:

(a) The intensity maximum of the interference pattern appears when

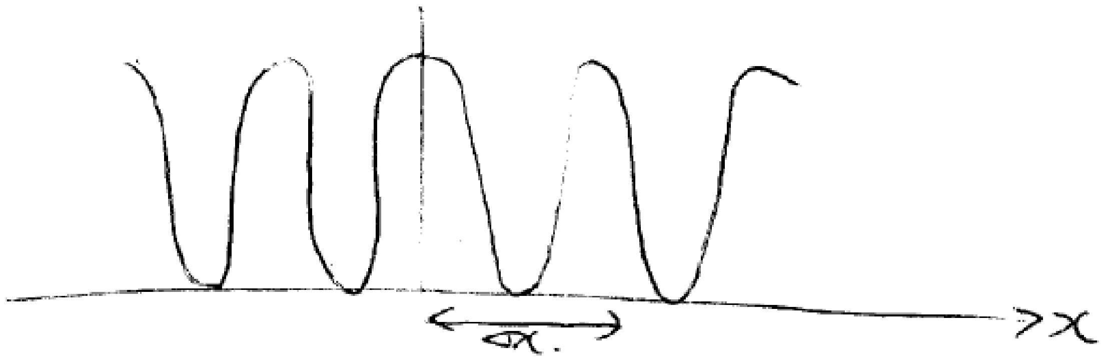
$$\delta = \frac{2\pi}{\lambda} d \frac{x}{D} = 2m\pi \quad => \quad x = \frac{m\lambda D}{d}$$

in which d is the separation between the slits, D is the distance from the double slits to the observing screen, P_o.

The separation of the adjacent intensity maximum on P_o is then

$$\Delta x = \frac{\lambda D}{d}.$$

So the plot of the interference pattern is shown as follow:



(b) As $\Delta x = \frac{\lambda D}{d}$, $\lambda = \frac{\Delta x d}{D} = \frac{5mm \cdot 0.8mm}{1.6m} = 2.5\mu m$.

(c) When one of the slits is covered by the film of transparent material, the phase difference is now $\delta = \frac{2\pi}{\lambda}(d\frac{x}{D} + nt)$, in which n is the refractive index of the material, 1.4. and t is the thickness of the film.

Before the film is there, the zeroth order is seen when $\delta = \frac{2\pi}{\lambda} d \frac{x}{D} = 0$, or

$x=0$; and now it is seen when $\delta = \frac{2\pi}{\lambda}(d\frac{x}{D} + nt) = 0$, or $x = -\frac{ntD}{d}$.

The zeroth order maximum is shifted by

$$\frac{ntD}{d} = 2.2\Delta x = 2.2\frac{\lambda D}{d}.$$

So the thickness of the film is $t = 2.2\frac{\lambda}{n} = 2.2 \times \frac{2.5\mu m}{1.4} = 3.93\mu m$.

(d) The maximum of 450nm occurs at

$$x_{1M} = \frac{m_1 \lambda_1 D}{d};$$

The minimum of this wavelength occurs at $x_{1m} = \frac{(m_1 + 0.5)\lambda_1 D}{d}$; while the maximum of 600nm occurs at

$$x_{2M} = \frac{m_2 \lambda_2 D}{d};$$

The minimum of this wavelength occurs at $x_{2m} = \frac{(m_2 + 0.5)\lambda_2 D}{d}$.

For a maximum of one wavelength falls exactly on a minimum of the other,

$$\frac{m_1 \lambda_1 D}{d} = \frac{(m_2 + 0.5)\lambda_2 D}{d} \quad \text{or} \quad \frac{m_2 \lambda_2 D}{d} = \frac{(m_1 + 0.5)\lambda_1 D}{d}$$

So $\frac{m_1}{(m_2 + 0.5)} = \frac{\lambda_2}{\lambda_1} = \frac{4}{3}$ (1) or $\frac{m_2}{(m_1 + 0.5)} = \frac{\lambda_1}{\lambda_2} = \frac{3}{4}$ (2)

The minimum integers for (1) to be valid are $m_1=2$, $m_2 = 1$, and no integer values will make (2) valid.

Therefore, the 2^{nd} order maximum of 450nm will fall exactly on the 1^{st} order minimum of 600nm.

5. ***Michelson Interferometer***

(a) How far must the movable mirror of a Michelson interferometer be displaced for 2500 fringes of the red cadmium line (6438 Å) to cross the center of the field of view?

(b) If the mirror of a Michelson interferometer is moved 1.0 mm, how many fringes of the blue cadmium line (4799.92 Å) will be counted crossing the field of view?

Solution:

(a) Whenever the movable mirror of a Michelson interferometer moves by $\lambda/2$, one fringe line crosses the center field of view.

So the movable mirror is displaced by $m\lambda/2 = 2500 \times 643.8\text{nm}/2 = 0.805\text{mm}$ for 2500 fringes to cross the center field of view.

(b) For blue cadmium line, when the mirror is moved 1.0mm, the number of fringes crossing the field of view is then

$m = 1.0\text{mm}/(\lambda/2) = 2.0\text{mm} / 479.992\text{nm} = 4167.$

6. ***Sagnac interferometer.*** A HeNe laser is used in a Sagnac interferometer. What area would be necessary so that a rotational velocity of 3 rad/s would correspond to 1 fringe shift?

Solution: As the number of fringes shift is $N = \frac{4A\Omega}{c\lambda}$,

The necessary area then is $A = \frac{Nc\lambda}{4\Omega} = \frac{3\times 10^8 m/s \times 632.8nm}{4\times 3rad/s} = 15.82m^2$.

7. ***Newton Rings.*** In an experiment involving Newton's rings, the diameters of the fifth and fifteenth bright rings formed by sodium yellow light (5889.95Å) are measured to be 2.303 and 4.134 mm, respectively. Calculate the radius of curvature of the convex glass surface.

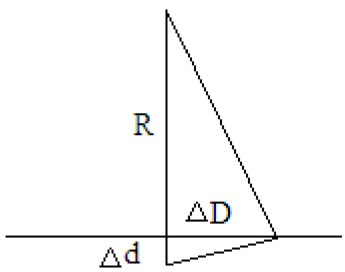
Solution: The bright rings occurs when $2nd + \lambda/2 = m\lambda$. The first bright ring occurs when m = 1. So the fifth and fifteenth bright rings correspond to m = 5 and m = 15, respectively.

So $d_1 = \frac{(5-0.5)\lambda}{2} = \frac{4.5\times 588.995nm}{2} = 1.325\mu m$

and $d_2 = \frac{(15-0.5)\lambda}{2} = \frac{14.5\times 588.995nm}{2} = 4.27\mu m$.

From the geometric relation(see the figure below) $\frac{\Delta D}{R} = \frac{\Delta d}{\Delta D}$, so the radius of curvature of the convex glass surface is

$$R = \frac{\Delta D^2}{\Delta d} = \frac{(4.134mm - 2.303mm)^2}{4.27\mu m - 1.325\mu m} = 1.138m.$$



8. ***Anti-Reflective Coating***

Design an anti-reflective coating for light of wavelength 950 nm to place on top of GaAs (n=3.6). Explain your design process. Specifically, what are the criteria which must be met to ensure zero reflected intensity?

Solution: When designing an AR coating, you choose two things: the refractive index and the thickness of the coating. You choose the refractive index so that the intensity of the two reflected beams are the same:

$$n = \sqrt{n_1 n_2} = \sqrt{1 \times 3.6} = 1.9 .$$

You choose the thickness so that the phase shift between the two reflected beams is a quarter wavelength:

$$nd = \frac{\lambda}{4}$$

So the film thickness is $d = \frac{950nm}{4 \times 1.9} = 125nm$