Lecture 19
SCALAR DIFFRACTION THEORY

[Reading assignment: Hect 10.2.4-10.2.6,10.2.8, 11.3.3]

Scalar Electromagnetic theory:

$$\boxed{u(P,t) = Re[U(P)e^{-j\omega t}]}$$ monochromatic wave

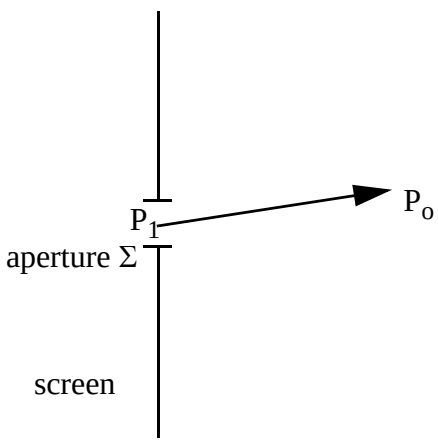
P : position t : time $\omega = 2\pi\nu$: optical frequency

u(P, t) represents the *E* or *H* field strength for a particular transverse polarization component

U(P) : represents the complex field amplitude

$U(P) = u(P)e^{-j\phi(P)}$ $u(P)$: real

<u>Diffraction:</u>



<u>Approximations:</u>

1. We impose the boundary condition on *U*, that *U* = 0 on the screen.
2. The field in the aperture Σ is not affected by the presence of the screen.

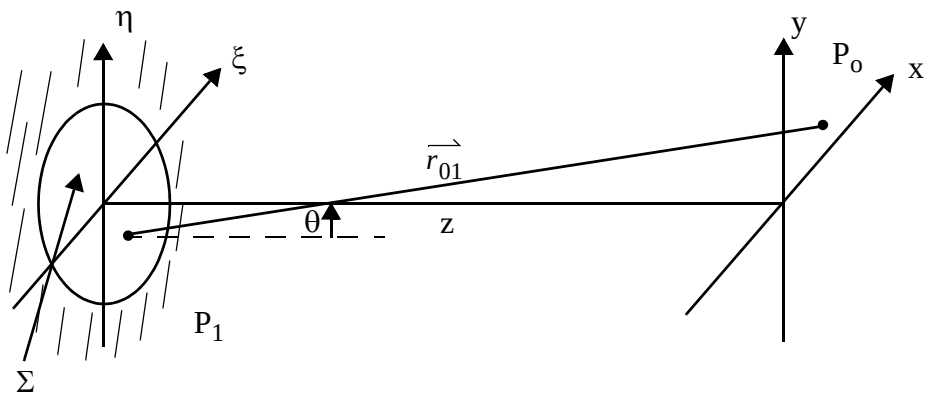
$$U(P_o) = \frac{1}{j\lambda}\iint_{\Sigma} U(P_1) \underbrace{\frac{\exp(jkr_{01})}{r_{01}} ds}_{\text{expanding spherical}}$$

$[r_{01} \gg \lambda]$

This equation expresses the Huygens-Fresnel principle: The observed field is expressed as a superposition of point sources in the aperture, with a weighting factor $\boxed{\dfrac{U(P_1)}{j\lambda}}$.

<u>Fresnel approximation</u>

Huygens-Fresnel integral in rectangular coordinates:



$$\boxed{r_{01} = [z^2 + (x-\xi)^2 + (y-\eta)^2]^{1/2}}$$

The Fresnel approximation involves setting: $r_{01} \cong z$ in the denominator, and

$$r_{01} \cong z\left[1 + \frac{1}{2}\frac{(x-\xi)^2}{z^2} + \frac{1}{2}\frac{(y-\eta)^2}{z^2}\right] \quad \text{in exponent}$$

This is equivalent to the paraxial approximation in ray optics.

$$U(x,y) = \frac{\exp(jkz)}{j\lambda z}\int\int_{-\infty}^{\infty} d\xi d\eta\, U(\xi,\eta)\exp\left\{\frac{jk}{2z}[(x-\xi)^2 + (y-\eta)^2]\right\} \qquad \text{(A)}$$

Let's examine the validity of the Fresnel approximation in the Fresnel integral. The next higher order term in exponent must be small compared to 1. So the valid range of the Fresnel approximation is:

$$z^3 \gg \frac{\pi}{4\lambda}[(x-\xi)^2 + (y-\eta)^2]^2_{max}$$

For field sizes of 1 cm, $\lambda = 0.5\mu m$, we find $z \gg 25$ cm.

Actually we should look at the effect on the total integral. Upon closer analysis, it is found that the Fresnel approximation holds for a much closer z. This is referred to as the "near-field region".

Farther out in z, we can approximate the quadratic phase as flat

$$z \gg \frac{k(\xi^2 + \eta^2)_{max}}{2}$$

This region is referred to as the "far-field" or Fraunhofer region.

$$U(x, y) = \frac{e^{jkz} e^{j\frac{k}{2z}(x^2 + y^2)}}{j\lambda z} \underbrace{\iint d\xi d\eta\, U(\xi, \eta) \exp\left[-j\frac{2\pi}{\lambda z}(x\xi + y\eta)\right]}_{F\{U(\xi,\eta)\}\big|_{f_x = \frac{x}{\lambda z},\, f_y = \frac{y}{\lambda z}}}$$

Now this is exactly the Fourier transform of the aperture distribution with

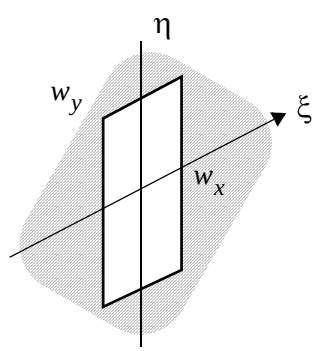
$$\boxed{f_x = \frac{x}{\lambda z}} \qquad \boxed{f_y = \frac{y}{\lambda z}}$$

The Fraunhofer region is farther out. For the field size of 1 cm, and $\lambda = 0.5\mu m$, we find the valid range of $z \gg 150$ meters!

Again, examining the full integral, Fraunhofer is actually accurate and usable to much closer distances.

<u>Examples</u>

A rectangular aperture, illuminated by a normally incident plane wave:



$$t_A = rect\left(\frac{\xi}{2w_x}\right) rect\left(\frac{\eta}{2w_y}\right)$$

With plane wave illumination, we have: $U(\xi, \eta) = t_A(\xi, \eta)$

$$\therefore \qquad U(x, y, z) = \frac{e^{jkz} e^{j\frac{k}{2z}(x^2 + y^2)}}{j\lambda z} F[U]\Big|_{\substack{f_x = \frac{x}{\lambda z} \\ f_y = \frac{y}{\lambda z}}}$$

$$= \frac{e^{jk\left[z + \frac{x^2+y^2}{z}\right]}}{j\lambda z} \, A\,\mathrm{sinc}\left(\frac{2w_x x}{\lambda z}\right) \mathrm{sinc}\left(\frac{2w_y y}{\lambda z}\right)$$

$$A \equiv 4 w_x w_y$$

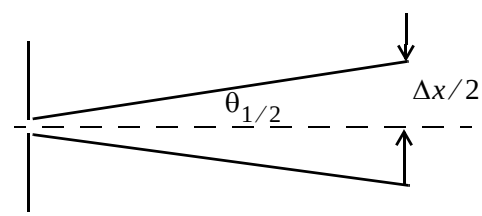
Recall $\boxed{\mathrm{sinc}(x) \equiv \frac{\sin \pi x}{\pi x}}$. The observable is intensity $I = |U|^2$.

$$I = \frac{A^2}{\lambda^2 z^2} \mathrm{sinc}^2\left(\frac{2w_x x}{\lambda z}\right) \mathrm{sinc}^2\left(\frac{2w_y y}{\lambda z}\right)$$

The width of the central lobe of the diffraction pattern is

$$\boxed{\Delta x = \frac{\lambda z}{w_x}}$$

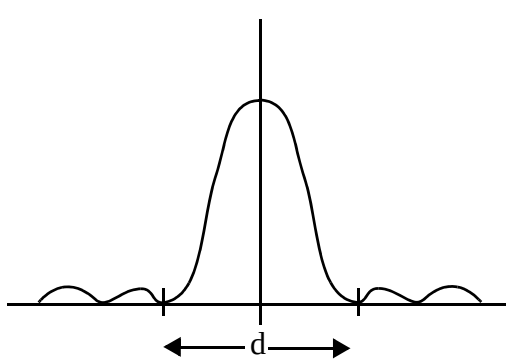
The diffraction half angle $\theta_{1/2} \cong \frac{\Delta x}{2} = \frac{\lambda}{2w_x}$

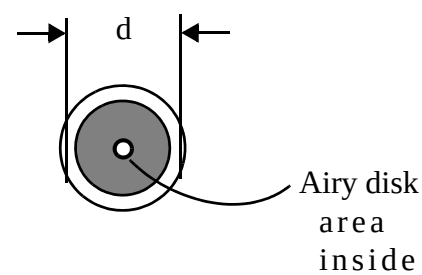


For a circular aperture with radius w : $t_A = circ\left(\frac{q}{w}\right)$ $\quad q^2 \equiv \xi^2 + \eta^2$ radial coordinates

In circular coordinates, we use the Fourier - Bessel transform: $B\{U(q)\}$ gives immediately:

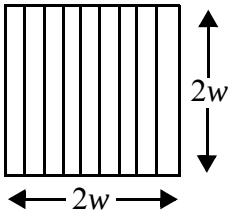
$$I(r) = \left(\frac{A}{\lambda z}\right)^2 \left[2\frac{J_1(kwr/z)}{kw(r/z)}\right]^2 \quad \text{“Airy pattern”}$$





$$\boxed{d = 1.22\frac{\lambda z}{w}}$$ diameter of Airy disk

<u>Diffraction grating</u> (transmission)

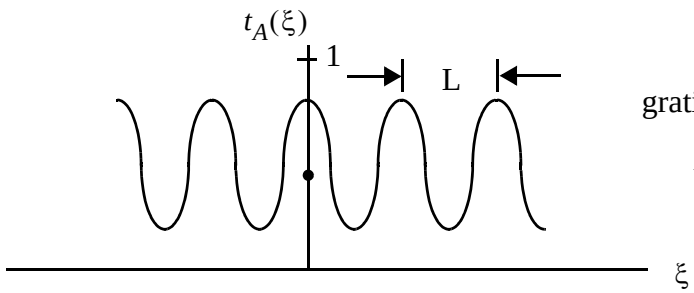


sinusoidal amplitude

$$t_A(\xi,\eta) = \left[\frac{1}{2} + \frac{m}{2}\cos(2\pi f_0 \xi)\right]\mathrm{rect}\left(\frac{\xi}{2w}\right)\mathrm{rect}\left(\frac{\eta}{2w}\right)$$

m: peak to peak amplitude change $\quad 0 \le m \le 1$

f_0: grating spatial frequency

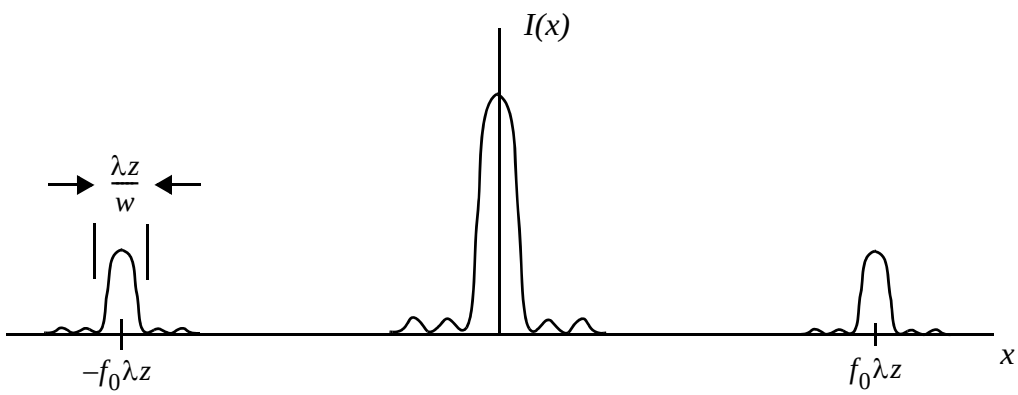


grating period

$$L = \frac{1}{f_0}$$

By convolution, the diffracted amplitude is

$$F\left[\frac{1}{2} + \frac{m}{2}\cos(2\pi f_0 \xi)\right] \otimes F\left[rect\frac{\xi}{2w} rect\frac{\eta}{2w}\right]$$



$$I(x,y) \cong \left(\frac{A}{2\lambda z}\right)^2 \mathrm{sinc}^2\frac{2wy}{\lambda z}\left\{\mathrm{sinc}^2\left(\frac{2wx}{\lambda z}\right) + \frac{m^2}{4}\mathrm{sinc}^2\left[\frac{2w}{\lambda z}(x + f_0\lambda z)\right] + \frac{m^2}{4}\mathrm{sinc}^2\left[\frac{2w}{\lambda z}(x - f_0\lambda z)\right]\right\}$$

We have neglected interference terms between orders.

Compared to the square aperture, which has the central peak with intensity I_o, we now have:

$$\frac{1}{4}I_0 \ : \ \text{zero order}$$

$$\frac{m^2}{16}I_0 \ : \ \pm 1 \text{ order}$$

The "resolving power" of the grating $R \equiv \dfrac{\text{peak separation}}{\text{peak width}}$

$$R = \frac{f_0 \lambda z}{\lambda z / w} = f_0 w = \frac{w}{L} = [\#\text{ grating periods}]$$