Lecture 19

SCALAR DIFFRACTION THEORY

[Reading assignment: Hect 10.2.4-10.2.6,10.2.8, 11.3.3]

Scalar Electromagnetic theory:

\[ u(P, t) = \text{Re}[U(P)e^{-j\omega t}] \]

monochromatic wave

\( P \): position \quad \omega = 2\pi \nu \): optical frequency

\( u(P, t) \) represents the \( E \) or \( H \) field strength for a particular transverse polarization component

\( U(P) \): represents the complex field amplitude

\[ U(P) = u(P)e^{-j\phi(P)} \quad u(P) : \text{real} \]

Diffraction:

\[ U(P_o) = \frac{1}{j\lambda} \int_{\Sigma} U(P_1) \exp\left(\frac{jkr_01}{\lambda}\right) ds \]

\([r_01 \gg \lambda]\) expanding spherical

Approximations:

1. We impose the boundary condition on \( U \), that \( U = 0 \) on the screen.
2. The field in the aperture \( \Sigma \) is not affected by the presence of the screen.
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This equation expresses the Huygens-Fresnel principle: The observed field is expressed as a superposition of point sources in the aperture, with a weighting factor \[ \frac{U(P_1)}{j\lambda}. \]

**Fresnel approximation**

Huygens-Fresnel integral in rectangular coordinates:

\[ r_{01} = \sqrt{z^2 + (x - \xi)^2 + (y - \eta)^2} \]

The Fresnel approximation involves setting: \( r_{01} \approx z \) in the denominator, and

\[ r_{01} \approx z \left[ 1 + \frac{1}{2} \left( \frac{x - \xi}{z^2} \right)^2 + \frac{1}{2} \left( \frac{y - \eta}{z^2} \right)^2 \right] \] in exponent

This is equivalent to the paraxial approximation in ray optics.

\[
U(x, y) = \frac{\exp(jkz)}{j\lambda z} \int \int_{-\infty}^{\infty} d\xi d\eta U(\xi, \eta) \exp \left[ \frac{jk}{2z} \left( \frac{(x - \xi)^2}{z^2} + \frac{(y - \eta)^2}{z^2} \right) \right]
\] (A)

Let’s examine the validity of the Fresnel approximation in the Fresnel integral. The next higher order term in exponent must be small compared to 1. So the valid range of the Fresnel approximation is:

\[ z^3 \gg \frac{\pi}{4\lambda} \left( \frac{(x - \xi)^2}{z^2} + \frac{(y - \eta)^2}{z^2} \right)_{\text{max}} \]

For field sizes of 1 cm, \( \lambda = 0.5 \mu m \), we find \( z \gg 25 \) cm.

Actually we should look at the effect on the total integral. Upon closer analysis, it is found that the Fresnel approximation holds for a much closer \( z \). This is referred to as the “near-field region”.

Farther out in \( z \), we can approximate the quadratic phase as flat.
The Fraunhofer region is farthest out. For the field size of 1 cm, and \( \lambda = 0.5 \mu m \), we find the valid range of \( z \approx 150 \) meters!

Again, examining the full integral, Fraunhofer is actually accurate and usable to much closer distances.

**Examples**

A rectangular aperture, illuminated by a normally incident plane wave:

With plane wave illumination, we have: \( U(\xi, \eta) = t_A(\xi, \eta) \)

\[
U(x, y, z) = \frac{e^{jkz} e^{j\frac{k}{2z}(x^2 + y^2)}}{j\lambda z F[U]} \bigg| _{f_x = \frac{x}{\lambda z}, f_y = \frac{y}{\lambda z}}
\]
Recall the observable is intensity. The width of the central lobe of the diffraction pattern is

\[ \Delta x = \frac{\lambda z}{w_x} \]

The diffraction half angle \( \theta_{1/2} \) is

\[ \theta_{1/2} = \frac{\lambda}{2w_x} \]

For a circular aperture with radius \( w \):

\[ t_A = \text{circ} \left( \frac{q}{w} \right) \]

In circular coordinates, we use the Fourier-Bessel transform: \( B \{ U(q) \} \) gives immediately:

\[ I(r) = \left( \frac{A}{\lambda z} \right)^2 \left[ \frac{J_1(kw r/z)}{kw(r/z)} \right]^2 \]

“Airy pattern”

\[ d = 1.22 \frac{\lambda z}{w} \]

diameter of Airy disk
Diffraction grating (transmission)

\[ t_A(\xi, \eta) = \left[ \frac{1}{2} + \frac{m}{2} \cos(2\pi f_0 \xi) \right] \text{rect} \left( \frac{\xi}{2w} \right) \text{rect} \left( \frac{\eta}{2w} \right) \]

\( m \): peak to peak amplitude change \( 0 \leq m \leq 1 \)

\( f_0 \): grating spatial frequency

\[ L = \frac{1}{f_0} \]

By convolution, the diffracted amplitude is

\[ F \left[ \frac{1}{2} + \frac{m}{2} \cos(2\pi f_0 \xi) \right] \otimes F \left[ \text{rect} \left( \frac{\xi}{2w} \right) \text{rect} \left( \frac{\eta}{2w} \right) \right] \]

\[ I(x) \]

\[ I(x, y) \equiv \left( \frac{A}{2\lambda z} \right)^2 \frac{\text{sinc}^2 \left( \frac{2w y}{\lambda z} \right)}{\lambda z} \left( \text{sinc}^2 \left( \frac{2w x}{\lambda z} \right) + \frac{m^2}{4} \text{sinc}^2 \left( \frac{2w}{\lambda z} (x + f_0 \lambda z) \right) + \frac{m^2}{4} \text{sinc}^2 \left( \frac{2w}{\lambda z} (x - f_0 \lambda z) \right) \right) \]

We have neglected interference terms between orders.

Compared to the square aperture, which has the central peak with intensity \( I_o \), we now have:
The “resolving power” of the grating

\[
\frac{1}{4}I_0 : \text{zero order} \\
\frac{m^2}{16}I_0 : \pm 1 \text{ order}
\]

\[
R = \frac{\text{peak separation}}{\text{peak width}}
\]

\[
R = \frac{f_0 \lambda z}{\lambda z/w} = f_0 w = \frac{w}{L} = [\text{# grating periods}]
\]