Lecture 5

Thin Lens Model

We now construct our model for the thin lens in air. Lens index is $n_l$. We will find the imaging properties of the thin lens by using the previous results for a single spherical surface and applying them twice - once for each of the two surfaces of the lens.

Use Eq. (4.12) for first surface: $n = 1$, $n' = n_l$, $l \rightarrow d_1$, $l' \rightarrow d'_1$

\[
\frac{n_l}{d'_1} - \frac{1}{d_1} = \frac{n_l - 1}{R_1}
\]  
(5.1)

We get a virtual object at $d'_1$

Now consider the rays travelling inside the lens from the virtual object. Apply spherical surface law now to the second surface. This time $n' = 1$, $n = n_l$, $l \rightarrow d'_2$, $l' \rightarrow d_2$

\[
\frac{1}{d_2} - \frac{n_l}{d'_2} = \frac{1 - n_l}{R_2}
\]
(5.2)

The thin lens approximation is that the lens thickness is negligible, so that $d'_2 \approx d'_1$. Using this in Eq (5.1), then substituting in Eq. (5.2),

\[
\frac{1}{d_2} - \frac{1}{d_1} = \frac{1 - n_l}{R_2} + \frac{n_l - 1}{R_1}
\]
(5.3)

This is the Gaussian lens law, with the focal length identified as:

\[
\frac{1}{f} = (n_l - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right)
\]
(5.4)

This is called the lensmaker’s equation.
We conclude that a lens with 2 spherical surfaces satisfies the Gaussian lens law, but only under 2 important approximations

- Paraxial approximation
- Thin-lens approximation

**Thick lens or compound lens systems**

*[Reading assignment: Hecht 6.1]*

Any symmetric optical system consisting of lenses and spaces can be generalized.

Light rays entering from the left, parallel to the optic axis, come to a focus, at the “second focal point”

Now, we take the rays entering the system and those emerging from the system and extend them. They intersect on a plane called the “Second principal plane”. Similarly, the first focus and “first principal plane” are defined for rays emerging from the system parallel to the axis, which all emanate from a point.

We define $f_2$ as the distance from the second principal plane to the second focal point. Similarly we define $f_1$ as the distance from the first principal plane to the first focal point. For a system immersed in air (same index on both sides), $f_1 = f_2$. 
With these definitions, the Gaussian lens law applies as follows:

\[ \frac{1}{d_2} = \frac{1}{f} + \frac{1}{d_1} \quad ; \quad m = \frac{d_2}{d_1} \quad ; \quad \bar{m} = m^2 \]

**Wave optics of lenses**

Set of rays parallel to axis

Plane Wave

\[ E(x,y,z) = E_0 \cos(kz - \omega t) \]

\[ k = \frac{2\pi}{\lambda} \]

\[ \omega = 2\pi f \]

Rays converging to a focus

Converging spherical wave
At a given z-plane, the spherical wave has constant phase around circles. The form of the spherical wave is 
\[ \cos \left( k \frac{x^2 + y^2}{2z_o} \right) \]
for a spherical wave converting to the point \( z_o \) on the axis. A lens modifies the wave front, for example from planar to spherical.

How does this happen?

**Optical path length:**

Optical waves travel more slowly in the glass since \( n > 1 \). In glass, the wave is delayed by an amount as if it travelled a distance \( nl \) in free space. If \( l = l(x,y) \) [or \( n = n(x,y) \)] then the delay varies with \( x, y \) so the wavefront gets distorted.

We can analyze the lens in terms of its phase-delay. The light propagates in the glass as 
\[ \cos(\text{knz}) = \cos\phi \]
where \( \phi = \text{k}nz \) is the phase delay.

In propagating from plane \( P_1 \) to \( P_2 \), the light travels a distance \( \Delta = \Delta_1 + \Delta_2 \) in the glass and a distance \( \Delta_0 - \Delta \) in air, where \( \Delta_0 \) is the thickness at the thickest part of the lens. The phase delay depends on \( x, y \):

\[ \phi(x, y) = k\text{n}\Delta(x,y) + k[\Delta_0 - \Delta(x,y)] \]

\[ = k\Delta_0 + k(n - 1)\Delta(x,y) \]
We can calculate $\Delta$, assuming spherical surfaces. Recall the sign convention for the surface radii:

- Positive radius
- Negative radius

From this diagram, we can readily obtain

$$\Delta(x, y) = \Delta_0 - \left[ R_1 - \sqrt{R_1^2 - x^2 - y^2} \right] + \left[ R_2 - \sqrt{R_2^2 - x^2 - y^2} \right]$$

$$= \Delta_0 - R_1 \left[ 1 - \frac{1}{\sqrt{1 - \left( \frac{x^2 + y^2}{R_{1, 2}} \right)}} \right] + R_2 \left[ 1 - \frac{1}{\sqrt{1 - \left( \frac{x^2 + y^2}{R_{1, 2}} \right)}} \right]$$

In the paraxial approximation $(x^2 + y^2) \ll R_{1, 2}^2$, so

$$\frac{1}{\sqrt{1 - \left( \frac{x^2 + y^2}{R_{1, 2}} \right)}} \approx 1 - \frac{1}{2} \left( \frac{x^2 + y^2}{R_{1, 2}} \right)$$

Thus

$$\Delta \cong \Delta_0 - \left( \frac{x^2 + y^2}{2} \right) \left( \frac{1}{R_1} - \frac{1}{R_2} \right)$$

This gives a phase delay:

$$\phi(x, y) = k\Delta_0 + k(n - 1) \left[ \Delta_0 - \left( \frac{x^2 + y^2}{2} \right) \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \right]$$

Apart from the constant delay $kn\Delta_0$, the phase delay is:

$$\phi(x, y) = -k(n - 1) \left( \frac{x^2 + y^2}{2} \right) \left( \frac{1}{R_1} - \frac{1}{R_2} \right)$$

A plane wave incident on the lens has a constant phase. After passing through the lens, the phase is given above. This has the form of a spherical wave, converging to a point at a distance $f$, where
\[ \frac{1}{f} = (n - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \]

is the focal length of the lens. This expression is identical to what we found from the ray optics analysis.