Examples of Analysis of Discrete-Time LTI Systems Using Z Transform

Example 1: First-Order System

Difference Equation

\[ y[n] - \alpha y[n-1] = x[n], \alpha \text{ real} \]

Transfer Function

Taking the bilateral Z transform of the difference equation:

\[ Y(z) - \alpha z^{-1} Y(z) = X(z) \]

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \alpha z^{-1}} = \frac{z}{z - \alpha} \]

\( H(z) \) has a zero at \( c_1 = 0 \) and a pole at \( d_1 = \alpha \).

Impulse Response

Inverting \( H(z) \) and using the right-sided inverse since the system is causal:

\[ h[n] = \alpha^n u[n]. \]

Step Response

The step response satisfies:

\[ s[n] = u[n] * h[n]. \]

Taking the bilateral Z transform:

\[ S(z) = \frac{1}{1 - z^{-1}} H(z) = \frac{1}{(1 - z^{-1})(1 - \alpha z^{-1})} = \frac{A_1}{1 - z^{-1}} + \frac{A_2}{1 - \alpha z^{-1}}, \]

where we assume \( \alpha \neq 1 \).

\[ A_1 = (1 - z^{-1}) S(z) \bigg|_{z = 1} = \frac{1}{1 - \alpha} \]

\[ A_2 = (1 - \alpha z^{-1}) S(z) \bigg|_{z = \alpha} = \frac{1}{1 - \alpha^{-1}} \]
Inverting $S(z)$, using the right-sided inverse since the system is causal:

$$s[n] = \frac{1}{1-\alpha} \cdot u[n] + \frac{1}{1-\alpha^{-1}} \cdot \alpha^n u[n], \quad \alpha \neq 1.$$

**Frequency Response**

Since the system is causal, the impulse response $h[n]$ is right-sided, and $ROC_H = \{ z | |z| > |\alpha| \}$. When $|\alpha| < 1$, the pole lies inside the unit circle, and the ROC includes the unit circle. In this case, the system is stable, and the frequency response exists:

$$H(z) \big|_{z = e^{j\Omega}} = H(e^{j\Omega}) = \frac{1}{1 - \alpha e^{-j\Omega}} \quad |\alpha| < 1.$$
Example 2: Second-Order System

Difference Equation

\[ y[n] - 2r \cos \theta y[n-1] + r^2 y[n-2] = x[n], \quad r > 0, \quad 0 \leq \theta \leq \pi. \]

Transfer Function

Taking the bilateral Z transform of the difference equation:

\[ Y(z) - 2r \cos \theta z^{-1} Y(z) + r^2 z^{-2} Y(z) = X(z). \]

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 2r \cos \theta z^{-1} + r^2 z^{-2}}. \]

\[ H(z) \] has two zeros at \( c_1 = 0 \) and poles at \( d_\frac{1}{2} = re^{\pm j\theta} \).

**Case (a): \( \theta = 0 \) (Critically Damped)**

**Poles**

When \( \theta = 0 \), \( d_1 = d_2 = r \), i.e., there is a single pole of multiplicity two.

**Impulse Response**

\[ h[n] = (n+1)r^n u[n]. \]

**Case (b): \( \theta = \pi \) (Underdamped)**

**Poles**

When \( \theta = \pi \), \( d_1 = d_2 = -r \), i.e., there is a single pole of multiplicity two.

**Impulse Response**

The mathematics is identical to case (a), but with the substitution \( r \rightarrow -r \):

\[ H(z) = \frac{1}{1 + 2rz^{-1} + r^2 z^{-2}} = \frac{1}{(1 + rz^{-1})^2}. \]

\[ h[n] = (n+1)(-r)^n u[n]. \]

**Case (c): \( 0 < \theta < \pi \) (Underdamped)**

**Poles**

When \( 0 < \theta < \pi \), \( d_\frac{1}{2} = re^{\pm j\theta} \), i.e., there are two distinct, complex-conjugate poles.

**Impulse Response**

In this case, it is necessary to do a partial-fraction expansion.
Examples of Analysis of Discrete-Time LTI Systems Using Z Transform

Choosing the right-sided inverse:

\[ H(z) = \frac{1}{1 - 2r \cos \theta z^{-1} + r^2 z^{-2}} = \frac{1}{(1 - re^{j\theta} z^{-1})(1 - re^{-j\theta} z^{-1})} = \frac{A_1}{1 - re^{j\theta} z^{-1}} + \frac{A_2}{1 - re^{-j\theta} z^{-1}} \]

\[ A_1 = (1 - re^{j\theta} z^{-1})H(z) \bigg|_{z = re^{j\theta}} = \frac{1}{1 - e^{j2\theta}} = \frac{e^{j\theta}}{2jsin\theta} \]

\[ A_2 = (1 - re^{-j\theta} z^{-1})H(z) \bigg|_{z = re^{-j\theta}} = \frac{1}{1 - e^{-j2\theta}} = \frac{-e^{-j\theta}}{2jsin\theta} \]

\[ H(z) = \frac{1}{2jsin\theta} \left[ \frac{e^{j\theta}}{1 - re^{j\theta} z^{-1}} - \frac{e^{-j\theta}}{1 - re^{-j\theta} z^{-1}} \right] \]

Choosing the right-sided inverse:

\[ h[n] = \frac{1}{2jsin\theta} \left[ e^{j\theta} (re^{j\theta})^n - e^{-j\theta} (re^{-j\theta})^n \right] u[n] = \frac{1}{sin\theta} r^n \sin((n + 1)\theta) u[n] \]

**Frequency Response (For Cases (a), (b) and (c))**

Since the system is causal, \( ROC_H = \{ z | |z| > r \} \). When \( 0 < r < 1 \), both poles lie inside the unit circle, and the ROC includes the unit circle. In this case, the system is stable, and the frequency response exists:

\[ H(z) \bigg|_{z = e^{j\Omega}} = H(e^{j\Omega}) = \frac{1}{1 - 2r \cos \theta e^{-j\Omega} + r^2 e^{-j2\Omega}}, \quad 0 < r < 1. \]